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Sombor indices - back to geometry

Ivan Gutman

Faculty of Science, University of Kragujevac, Kragujevac, Serbia.; gutman@kg.ac.rs

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Abstract: The Sombor index (SO) is a vertex-degree-based graph invariant, defined as the sum over all pairs of adjacent vertices of $\sqrt{d_i^2 + d_j^2}$, where d_i is the degree of the i-th vertex. It has been conceived using geometric considerations. Numerous researches of SO that followed, ignored its geometric origin. We now show that geometry-based reasonings reveal the geometric background of several classical topological indices (Zagreb, Albertson) and lead to a series of new SO-like degree-based graph invariants.

Keywords: Sombor index; Degree (of vertex); Vertex-degree-based graph invariant.

MSC: 05C07; 05C09.

1. Introduction

In this paper, we are concerned with a geometric interpretation of vertex-degree-based topological indices, extending the considerations from Ref. [1]. Let G be a simple graph with vertex set $\mathbf{V}(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $\mathbf{E}(G)$. The degree (= number of first neighbors) of the vertex $v_i \in \mathbf{V}(G)$ will be denoted by d_i. If the vertices v_i and v_j are adjacent, then the edge connecting them is labeled by ij.

In the mathematical and chemical literature, several dozens of vertex–degree–based (VDB) graph invariants (usually referred to as "topological indices") have been introduced and extensively studied [1–4]. Their general formula is

$$TI(G) = \sum_{ij \in \mathbf{E}(G)} F(d_i, d_j)\,, \tag{1}$$

where $F(x,y)$ is some function with the property $F(x,y) = F(y,x)$.

Some of these VDB indices, mentioned in the later part of the paper, are the first and second Zagreb index [5,6]

$$M_1(G) = \sum_{ij \in \mathbf{E}(G)} \left(d_i + d_j\right), \qquad \text{and} \qquad M_2(G) = \sum_{ij \in \mathbf{E}(G)} d_i\, d_j\,, \tag{2}$$

the Albertson index [7]

$$Alb(G) = \sum_{ij \in \mathbf{E}(G)} \left|d_i - d_j\right|, \tag{3}$$

and the Sombor index [1]

$$SO(G) = \sum_{ij \in \mathbf{E}(G)} \sqrt{d_i^2 + d_j^2}\,. \tag{4}$$

In [1], the ordered pair (a,b), where $a = d_i$, $b = d_j$, $a \geq b$, is called the *degree-coordinate* of the edge $ij \in \mathbf{E}(G)$, and represented by a *degree-point* in a two-dimensional coordinate system, see Figure 1. The point with coordinates (b,a) is then referred to as the *dual-degree-point* pertaining to the edge ij.

The distance between the degree-point of the edge ij (denoted by A in Figure 1), and the origin (O), assuming an Euclidean metrics, is $\sqrt{d_i^2 + d_j^2}$. Summing this over all edges of the graph G, one obtains the right–hand side of Eq. (4), i.e., the Sombor index [1].

We indicate this transformation as

$$\sum_{ij \in \mathbf{E}(G)} dist(AO) = SO(G)\,. \tag{5}$$

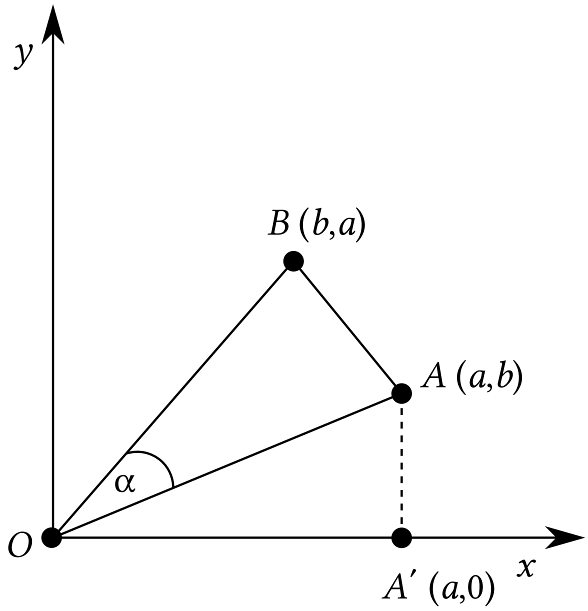

Figure 1. A geometric representation of an edge $e = ij$ of the graph G, connecting vertices of degree $d_i = a$ and $d_j = b$ $(a > b)$. The point A represents the edge ij, whereas the point B is the respective dual. O denotes the origin of the coordinate system.

In fact, the Sombor index was actually conceived by using the above geometric considerations [1]. This VDB topological index soon attracted much attention, and its numerous mathematical [8–14] and chemical [15–19] applications have been established. Yet, with a single exception [20], the geometry-based features of the Sombor index were ignored, and only its algebraic and combinatorial aspects were in focus of interest.

In this paper we intend to point out the great variety of Sombor-index-like VDB graph invariants that can be constructed by means of geometric arguments.

First of all, we recall that in [1], it was shown that the distance between the degree-point and its dual (points A and B in Figure 1), is equal to $\sqrt{2}\,|d_i - d_j|$, which after adding over all edges yields the Albertson index, i.e., $\sqrt{2}\,Alb(G)$, cf. Eq. (3). In analogy to formula (5), we write this as:

$$\sum_{ij\in \mathbf{E}(G)} dist(AB) = \sqrt{2}\,Alb(G)\,. \tag{6}$$

Let A' be the projection of the point A on the x-axis, see Fig. 1. Consider the triangle $AA'O$. Then, bearing in mind Eqs (2) and (4), we straightforwardly conclude that

$$\sum_{ij\in \mathbf{E}(G)} perim(AA'O) = \sum_{ij\in \mathbf{E}(G)} \left[\sqrt{d_i^2 + d_j^2} + d_i + d_j\right] = SO(G) + M_1(G)\,,$$

and

$$\sum_{ij\in \mathbf{E}(G)} area(AA'O) = \sum_{ij\in \mathbf{E}(G)} \frac{1}{2}\,d_i\,d_j = \frac{1}{2}\,M_2(G)\,,$$

where $perim(X)$ and $area(X)$ stand for the perimeter and area of the geometric object X.

Thus, a few classical VDB topological indices have a geometry-based interpretation.

2. More Sombor-index-like graph invariants

In this section we consider the triangle formed by the degree-point, dual-degree-point, and the origin of the coordinate system, points A, B, and O in Figure 1. From this triangle, we generate a number of new Sombor-index-like VDB invariants, denoted below by $SO_1, SO_2, \ldots, SO_6$.

First note that by bearing in mind Eqs. (5) and (6), we immediately get

$$\sum_{ij\in\mathbf{E}(G)} perim(ABO) = 2\,SO(G) + \sqrt{2}\,Alb(G)\,.$$

Using elementary geometry (Heron's formula), knowing the distance between the points A and O (same as between B and O), cf. Eq. (5), as well as between A and B, cf. Eq. (6), we obtain

$$area(ABO) = \frac{1}{2}\left(a^2 - b^2\right),$$

from which it follows

$$\sum_{ij\in\mathbf{E}(G)} area(ABO) = \sum_{ij\in\mathbf{E}(G)} \frac{1}{2}\left|d_i^2 - d_j^2\right| := SO_1(G)\,.$$

Let α be the angle between the lines OA and OB, see Figure 1. Using the classical formula

$$area(ABO) = \frac{1}{2}\,dist(OA)\,dist(OB)\,\sin\alpha\,,$$

we get

$$\frac{1}{2}\left|d_i^2 - d_j^2\right| = \frac{1}{2}\left(\sqrt{d_i^2 + d_j^2}\right)^2 \sin\alpha\,,$$

from which

$$\sum_{ij\in\mathbf{E}(G)} \sin\alpha = \sum_{ij\in\mathbf{E}(G)} \left|\frac{d_i^2 - d_j^2}{d_i^2 + d_j^2}\right| := SO_2(G)\,.$$

Denote by Γ_c the circumcircle (circumscribed circle) on the triangle ABO. The respective incircle (inscribed circle) will be denoted by Γ_i. Using elementary, yet somewhat lengthy, geometric reasoning, it can be shown that

$$perim(\Gamma_c) = \sqrt{2}\,\frac{a^2+b^2}{a+b}\,\pi\,, \tag{7}$$

$$area(\Gamma_c) = \frac{1}{2}\left(\frac{a^2+b^2}{a+b}\right)^2\pi\,, \tag{8}$$

$$perim(\Gamma_i) = \frac{2(a^2-b^2)}{\sqrt{2}+2\sqrt{a^2+b^2}}\,\pi\,, \tag{9}$$

$$area(\Gamma_i) = \left[\frac{a^2-b^2}{\sqrt{2}+2\sqrt{a^2+b^2}}\right]^2\pi\,. \tag{10}$$

In connection with Eqs. (9) and (10), note that if the center of the incircle has coordinates (γ,γ) and its distance from the origin is $\sqrt{2}\,\gamma$, then

$$radius(\Gamma_i) = \frac{(a-b)\gamma}{\sqrt{a^2+b^2}} \qquad \text{and} \qquad radius(\Gamma_i) + \sqrt{2}\,\gamma = \frac{a+b}{\sqrt{2}}\,.$$

The formulas (7)–(10) make it possible to introduce the following Sombor-index-like VDB invariants

$$SO_3(G) = \sum_{ij\in\mathbf{E}(G)} perim(\Gamma_c) = \sum_{ij\in\mathbf{E}(G)} \sqrt{2}\,\frac{d_i^2+d_j^2}{d_i+d_j}\,\pi\,,$$

$$SO_4(G) = \sum_{ij\in\mathbf{E}(G)} area(\Gamma_c) = \frac{1}{2}\sum_{ij\in\mathbf{E}(G)} \left(\frac{d_i^2+d_j^2}{d_i+d_j}\right)^2\pi\,,$$

$$SO_5 = \sum_{ij\in \mathbf{E}(G)} perim(\Gamma_i) = \sum_{ij\in \mathbf{E}(G)} \frac{2\,|d_i^2 - d_j^2|}{\sqrt{2} + 2\sqrt{d_i^2 + d_j^2}}\,\pi\,,$$

$$SO_6 = \sum_{ij\in \mathbf{E}(G)} area(\Gamma_i) = \sum_{ij\in \mathbf{E}(G)} \left[\frac{d_i^2 - d_j^2}{\sqrt{2} + 2\sqrt{d_i^2 + d_j^2}}\right]^2 \pi\,.$$

It would be interesting to examine the properties of these geometry-based topological indices, and see if these are usable in applications. What is immediately evident is that SO_1, SO_2, SO_5, and SO_6 are suitable for measuring graph irregularity [21], i.e., they are equal to zero if and only the underlying graph is regular.

Conflicts of Interest: "The author declares no conflict of interest."

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