Asian Research Journal of Mathematics

7(3): 1-22, 2017; Article no.ARJOM.37471
ISSN: 2456-477X

A Simple SEIR Mathematical Model of Malaria Transmission

Mojeeb AL-Rahman EL-Nor Osman[1,2*], Isaac Kwasi Adu[1,3] and Cuihong Yang[1]

[1]School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P.R. China.
[2]Department of Mathematics and Computer Science, International University of Africa, P.O.Box 2469, Khartoum, Sudan.
[3]Department of Mathematics, Valley View University, Techiman Campus, P.O.Box 183 B/A, Ghana.

Authors' contributions

This work was carried out in collaboration between all authors. Authors MARENO and IKA assisted in developing the model equations, writing of the draft, numerical simulations and review of the final draft. Author CY reviewed the final draft. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/37471
Editor(s):
(1) Ruben Dario Ortiz Ortiz, Professor, Facultad de Ciencias Exactas y Naturales, Universidad de Cartagena, Colombia.
Reviewers:
(1) Hugo Cruz-Suárez, Benemérita Universidad Autónoma de Puebla, Mexico.
(2) Jagdish Prakash, University of Botswana, Botswana.
Complete Peer review History: http://www.sciencedomain.org/review-history/22060

Original Research Article

Received: 18th October 2017
Accepted: 3rd November 2017
Published: 27th November 2017

Abstract

We have studied an SEIR mathematical model, and applied it to malaria transmission. We discussed the existence and stability of the Disease-Free (DFE) and Endemic Equilibria (EE) of both models. The (DFE) was locally asymptotically stable if the reproduction number is less than one and unstable if the reproduction number is greater than one for SEIR and malaria transmission model. Numerical simulations using Matlab Software were conducted to confirm our analytic results. Our findings were that, Malaria may be controlled by reducing the contact rate between human and mosquito, reducing the infection rate between the human, the use of active malaria drugs, insecticides and mosquito treated nets can also help to reduce mosquitoes population and malaria transmission respectively.

Keywords: Mathematical model; reproduction number; disease-free equilibrium; endemic equilibrium; stability.

**Corresponding author: E-mail: mojeeb_osman@yahoo.com;*

1 Introduction

Mosquito-borne diseases, including malaria, transmitted to humans by mosquitoes, are big concerns in public health. Malaria is the fifth cause of death from infectious diseases worldwide (after respiratory infections, HIV/AIDS, diarrheal diseases, and tuberculosis), and the second leading cause of death from infectious diseases in Africa, after HIV/AIDS [1,2,3]. The first evidence of existence of malaria has been obtained from mosquitoes preserved in amber, dated back to about 30 million years ago [4]. Even though the disease has been investigated for hundreds of years it still remains a major public health problem with 91 countries. The global record of malaria in 2015 was 212 million new cases and 429000 deaths. Across Africa, millions of people still lack access to the tools they need to prevent and treat the disease [5]. No vaccines are available and an effective way to prevent malaria is to control mosquitoes [6]. Malaria has for many years been considered as a global issue, and many epidemiologists and other scientists invest their effort in learning the dynamics of malaria and to control its transmission. From interactions with those scientists, mathematicians have developed a significant and effective tool, namely mathematical models of malaria, giving an insight into the interaction between the host and vector population, the dynamics of malaria, how to control malaria transmission, and eventually how to eradicate it[7]. A lot of work has been done on the epidemiological models and malaria transmission that are related to this article include that of: Ngwa and Shu [8]. They developed and analyzed an SEIRS model to study the dynamics and transmission of malaria, involving variable human and mosquito populations. According to their results, there is a threshold parameter R_0 and the disease can persist if and only if $R_0 > 1$ and the (DFE) always exists and is locally stable if $R_0 < 1$, and unstable if $R_0 > 1$. Their model was also globally stable when $R_0 \leq 1$. They confirmed their results with numerical simulations. Their model provides a frame work for studying control strategies for the containment of malaria. Malaria model which is relevant to this work is the work of another Olaniyi and Obabiyi [9]. They used a system of seven-dimensional ODE'S to model the transmission of *Plasmodium falciparum* malaria between humans and mosquitoes with non-linear forces of infection in form of saturated incidence rates, these incidence rates produce antibodies in response to the presence of parasite causing malaria in both human and mosquito populations. They investigate the stability analysis of (DFE) and according to their results, (DFE) is asymptotically stable when $R_0 < 1$, and unstable when $R_0>1$. They also determined the existence of the unique (EE) under certain conditions, and their numerical simulation confirms the analytical result. Furthermore, Jia Li [10] developed an SEIR malaria model with stage-structured mosquitoes. They included metamorphic stages in the mosquito population and a simple stage mosquito population is introduced, were the mosquito population is divided into two classes namely, the aquatic stage in one class and all adults in the other class. According to their results the different dynamical behaviour of the models in their study, compared to other the bahaviour of most classical epidemiological models, and the possible occurrence of backward bifurcation make control of malaria more difficult. Moreover, Altaf Khan et al. [11] formulate an SEIR model with non-linear saturated incidence rate and temporary immunity, they divided the total into four subclasses susceptible, exposed, infected and recovered. They assumed that the total population is constant, and the new born babies are susceptible with no migration, they investigate the stability of disease- free and endemic equilibria. Their results shows that when $R_0 < 1$ the DFE is stable locally as well as globally and EE is stable locally as well as globally when $R_0 > 1$.and their theoretical results was justified by the numerical simulation. Shah and Gupta [12] Modeled the basic SEIR model and applied it to vector borne disease (malaria). They carried out the sensitivity analysis of the model using data from India. According to their results, the sensitivity analysis was very important, and it is the most sensitive aspect to be taken care of in their model.

In this paper, we use SEIR model based on Olaniyi and Obabiyi [9] and Shah and Gupta [12], and apply it to malaria transmission between mosquitoes and humans. We assume that recovered human individuals can enter into the susceptible class again, mosquitoes never enter the recovered class and the newborn's birth with infection can enter I compartment. We determine the stability analysis of the disease – free and endemic equilibria. The rest of the paper is organized as follows: In section 2, we present the SEIR model description and derived the basic reproduction number. Model analysis consisting of the stability analysis of disease-free and endemic equilibria is discussed in section 3. In section 4 we apply the SEIR model to malaria

transmission. We use numerical simulation to show the dynamical behaviour of our results in section 5. Section 6 is made up of discussion of our results. We ended the paper with a conclusion in section 7.

2 Mathematical Model

2.1 Model description and basic reproduction number

The Population of our model is divided into four compartments: Susceptible Humans (t) .

Exposed Humans $E(t)$, Infectious Humans $I(t)$ and Removed Humans $R(t)$. The interaction between the four compartments is shown in the schematic diagram in Fig. 1, below:

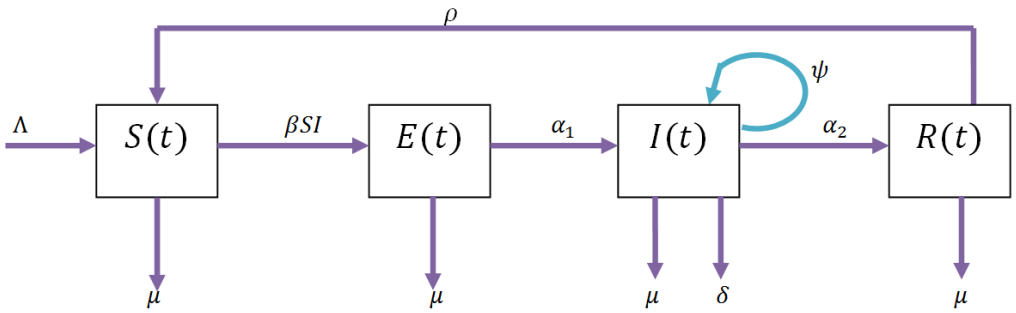

Fig. 1. Schematic diagram of malaria transmission

2.2 Model assumptions

The following assumptions were made in the model:

(i) The number of infected people increases at a rate proportional to both the number of infectious and the number of susceptible.
(ii) Humans moves from Exposed to Infectious compartments with progression rate α_1
(iii) The rate of removal of infectious to recovered compartment is proportional to the number of infectious Only.
(iv) A human can die at any stage by natural causes.
(v) Newborn's birth with infection can enter I compartment with rate.
(vi) Recovered humans can enter S compartment at rate ρ.

The model is given by the system of ODE's as:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta SI - \mu S + \rho R \\ \frac{dE}{dt} = \beta SI - (\alpha_1 + \mu)E \\ \frac{dI}{dt} = \alpha_1 E - (\alpha_2 + \mu + \delta)I + \psi I \\ \frac{dR}{dt} = \alpha_2 I - (\mu + \rho)R \end{cases} \tag{2.1}$$

with the initial conditions $S(0) = S_0, E(0) = E_0, I(0) = I_0$ and $R(0) = R_0$.

where, Λ is the recruitment rate of the population, β effective infection rate, μ is the natural death rate, δ is the disease induced death rate, $\alpha_1$is developing rate of exposed (humans) becoming infectious and $\alpha_2$is the recovered rate of humans. ψ is the Newborn's birth rate entered I compartment.

Table 1. Parameters description of SEIR model

Parameter	Parameter description
Λ	Input flow of the susceptible
β	Infection rate (effective infection rate)
α_1	Developing rate of exposed (humans)
α_2	Recover rate of humans
μ	Natural death rate
δ	induce death rate
ψ	Newborn's birth rate entered compartment I

We also consider the following equations:

$$N(t) = S(t) + E(t) + I(t) + R(t) \tag{2.2}$$

Then the derivative of $N(t)$ with respect to t is given by:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

$$\frac{dN}{dt} = \Lambda - \mu N - \alpha_2 I - (\delta - \beta)I$$

$$\frac{dN}{dt} \leq \Lambda - \mu N$$

It follows that:

$$\lim_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}. \tag{2.3}$$

Thus the feasible region of the system (2.1) is given by

$$\Gamma = \{(S, E, I, R): S + E + I + R \leq \frac{\Lambda}{\mu}, S > 0, E \geq 0, I \geq 0, R \geq 0\}$$

Is positively invariant. Next, we discuss the basic reproduction number of the system (2.1) by using the next generation matrix process [13]. It is easy to see that the system (2.1) has the disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$.

Let $X = (S, E, I, R)^T$, then system (1) can be written as

$$X' = F(X) - V(X),$$

Where:

$$F(X) = \begin{bmatrix} \beta IS \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{And} \quad V(X) = \begin{bmatrix} (\alpha_1 + \mu)E \\ -\alpha_1 E + (\alpha_2 + \mu + \delta)\mathrm{I} - \psi I \\ -\Lambda + \beta SI + \mu\mathrm{S} - \rho\mathrm{R} \\ -\alpha_2 I + (\mu + \rho)R \end{bmatrix}$$

The Jacobian matrices of $F(X)$ and $V(X)$ at the disease-free equilibrium, E_0 are respectively

$$DF(E_0) = \begin{bmatrix} 0 & \frac{\beta\Lambda}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, DV(E_0) = \begin{bmatrix} (\alpha_1+\mu) & 0 & 0 & 0 \\ -\alpha_1 & (\alpha_2+\mu+\delta)-\psi & 0 & 0 \\ 0 & \frac{\beta\Lambda}{\mu} & \mu & -\rho \\ 0 & -\alpha_2 & 0 & \mu+\rho \end{bmatrix}$$

Let: $\mathcal{F} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$ And $\mathcal{V} = \begin{bmatrix} \mathrm{V} & 0 \\ \mathcal{J}_1 & \mathcal{J}_2 \end{bmatrix}$

Where:

$$F = \begin{bmatrix} 0 & \frac{\beta\Lambda}{\mu} \\ 0 & 0 \end{bmatrix} \text{ And } V = \begin{bmatrix} (\boldsymbol{\alpha_1+\mu}) & 0 \\ -\alpha_1 & (\boldsymbol{\alpha_2+\mu+\delta})-\boldsymbol{\psi} \end{bmatrix}$$

The reproduction number is given by the spectral radius of FV^{-1} that is

$$R_0 = FV^{-1} = \begin{bmatrix} 0 & \frac{\beta\Lambda}{\mu} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(\alpha_1+\mu)} & 0 \\ \frac{\alpha_1}{(\alpha_1+\mu)((\alpha_2+\mu+\delta)-\psi)} & \frac{1}{(\alpha_2+\mu+\delta)-\psi} \end{bmatrix}$$

$$R_0 = \frac{\alpha_1\beta\Lambda}{\mu(\boldsymbol{\alpha_1+\mu})(\boldsymbol{\alpha_2+\mu+\delta-\psi})} \tag{2.4}$$

Theorem 1: The disease-free equilibrium $E_0\left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ of the system (2.1) is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

3 Model Analysis

3.1 Disease-free equilibrium

In this section, we investigate the local geometrical properties of the disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$ by considering the linearized system of ODE's (2.1), by taking the Jacobian matrix and obtained

$$J(S,E,I,R) = \begin{bmatrix} -\beta I-\mu & 0 & -\beta S & \rho \\ \beta I & -(\alpha_1+\mu) & \beta S & 0 \\ 0 & \alpha_1 & -(\delta+\mu+\alpha_2)+\psi & 0 \\ 0 & 0 & \alpha_2 & -(\mu+\rho) \end{bmatrix} \tag{3.1}$$

The local stability of the equilibrium may be determined from the Jacobian matrix (3.1). This implies that the Jacobian matrix for the drinking-free equilibrium is given by

$$J(E_0) = \begin{bmatrix} -\mu & 0 & -\beta\frac{\Lambda}{\mu} & \rho \\ 0 & -(\alpha_1+\mu) & \beta\frac{\Lambda}{\mu} & 0 \\ 0 & \alpha_1 & -(\delta+\mu+\alpha_2)+\psi & 0 \\ 0 & 0 & \alpha_2 & -(\mu+\rho) \end{bmatrix} \tag{3.2}$$

To find the eginvalues of the matrix (3.2) use determinant

$$|J(E_0) - \lambda I| = \begin{vmatrix} -\mu - \lambda & 0 & -\beta\frac{\Lambda}{\mu} & \rho \\ 0 & -(\alpha_1 + \mu) - \lambda & \beta\frac{\Lambda}{\mu} & 0 \\ 0 & \alpha_1 & -(\delta + \mu + \alpha_2) + \psi - \lambda & 0 \\ 0 & 0 & \alpha_2 & -(\mu + \rho) - \lambda \end{vmatrix} = 0 \quad (3.3)$$

The eigenvalues are given by:

Clearly $\lambda_1 = -\mu\,, \lambda_2 = -(\mu + \rho)$ are negatives and

$$\lambda^2 + a_1\lambda + a_2 = 0 \quad (3.4)$$

$$a_1 = 2\mu + \alpha_1 + \alpha_2 + \delta - \psi$$

$$a_2 = \alpha_1\delta + \alpha_1\mu + \alpha_1\alpha_2 + \mu\delta + \mu^2 + \mu\alpha_2 - \alpha_1\psi - \mu\psi - \frac{\alpha_1\beta\Lambda}{\mu}$$

Using the Routh-Hurwitz criterion [14,15], it can be seen that all the eigenvalues of the characteristic equation (3.4) have negative real part if and only if:

$$a_1 > 0,\ a_1a_2 > 0 \quad (3.5)$$

Theorem 2: E_0 is locally asymptotically stable if and only if inequalities (3.5) are satisfied.

3.2 Existence of endemic equilibrium

In this section, we consider a situation in which all the disease states coexist in the equilibrium. We denote $E^* = (S^*, E^*, I^*, R^*)$ as the endemic equilibrium of the system (2.1). We also obtain

$$S^* = \frac{(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{\beta\alpha_1} \quad , E^* = \frac{(\delta+\mu+\alpha_2-\psi)[\alpha_1\beta\Lambda(\mu+\rho)-\mu(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)]}{\alpha_1\beta[(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)-\alpha_1\alpha_2\rho]}$$

$$I^* = \frac{\alpha_1\beta\Lambda(\mu+\rho)-\mu(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{\beta[(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)-\alpha_1\alpha_2\rho]} \text{ and } R^* = \frac{\alpha_1\alpha_2\beta\Lambda-\mu\alpha_2(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{\beta[(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)-\alpha_1\alpha_2\rho]}$$

from system of ODE's (2.1) and linearized the same system to obtained:

$$J(S,E,I,R) = \begin{bmatrix} -\beta I - \mu & 0 & -\beta S & -\rho \\ \beta I & -(\alpha_1 + \mu) & \beta S & 0 \\ 0 & \alpha_1 & -(\delta + \mu + \alpha_2) + \psi & 0 \\ 0 & 0 & \alpha_2 & -(\mu + \rho) \end{bmatrix}$$

$J(E^*) =$

$$\begin{bmatrix} -[\frac{\alpha_1\beta\Lambda(\mu+\rho)-\mu(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{[(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)-\alpha_1\alpha_2\rho]}] - \mu & 0 & -[\frac{(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{\alpha_1}] & -\rho \\ [\frac{\alpha_1\beta\Lambda(\mu+\rho)-\mu(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{[(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)-\alpha_1\alpha_2\rho]}] & -(\alpha_1 + \mu) & [\frac{(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{\alpha_1}] & 0 \\ 0 & \alpha_1 & -(\delta + \mu + \alpha_2) + \psi & 0 \\ 0 & 0 & \alpha_2 & -(\mu + \rho) \end{bmatrix} \quad (3.6)$$

We determine the local stability of the positive equilibrium E^*, by using the following lemma.

Lemma 2 [10, 13]: Let H be a 4×4 real matrix. If $tr(H)$, $det\ (H)$ and $det\ (H^{[2]})$ are all negative, then all the eigenvalues of H have negative real part.

Definition 1 [16] Let B be a real $m \times m$ matrix. The second additive compound matrix of $B = (b_{ij})$ for $m = 4$ is defined as

$$B^{[2]} = \begin{bmatrix} b_{11}+b_{22} & b_{23} & b_{24} & -b_{13} & -b_{14} & 0 \\ b_{32} & b_{11}+b_{33} & b_{34} & b_{12} & 0 & -b_{14} \\ b_{42} & b_{43} & b_{11}+b_{44} & 0 & b_{12} & b_{13} \\ -b_{31} & b_{21} & 0 & b_{22}+b_{33} & b_{34} & -b_{24} \\ -b_{41} & 0 & b_{21} & b_{43} & b_{22}+b_{44} & b_{23} \\ 0 & -b_{41} & b_{31} & -b_{42} & b_{32} & b_{33}+b_{44} \end{bmatrix} \tag{3.7}$$

Theorem 3: The positive equilibrium E^* of the system (2.1) is locally asymptotically stable if $R_0 > 1$.

Proof: construct a second additive compound matrix $J^{[2]}(E^*)$of $J(E^*)$ and obtain

$$J^{[2]}(E^*) = \begin{bmatrix} J_0 & J_1 \\ J_2 & J_3 \end{bmatrix} \tag{3.8}$$

$$J_0 = \begin{bmatrix} -(\beta I^* + 2\mu + \alpha_1) & \beta S^* & 0 \\ \alpha_1 & -(\beta I^* + 2\mu + \alpha_2 + \delta - \psi) & 0 \\ 0 & \alpha_2 & -(\beta I^* + 2\mu + \rho) \end{bmatrix}$$

$$J_1 = \begin{bmatrix} \beta S^* & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 0 & -\beta S^* \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & \beta I^* & 0 \\ 0 & 0 & \beta I^* \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} -(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi) & 0 & 0 \\ \alpha_2 & -(\alpha_1 + 2\mu + \rho) & \beta S^* \\ 0 & \alpha_1 & -(\delta + 2\mu + \alpha_2 + \rho - \psi) \end{bmatrix}$$

Then from the above:

$$tr\big(J(E^*)\big) = -(\beta I^* + 3\mu + \alpha_1 + \rho + \delta + \alpha_2 - \psi)$$
$$= -\left(\frac{\alpha_1\beta\Lambda(\mu+\rho) - \mu(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)}{[(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi) - \alpha_1\alpha_2\rho]}\right) + 3\mu + \alpha_1 + (\rho + \delta + \alpha_2 - \psi)) < 0$$

If $\alpha_1\beta\Lambda(\mu+\rho) > \mu(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)$,

$(\mu+\rho)(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi) > \alpha_1\alpha_2\rho$ When $\rho + \delta + \alpha_2 > \psi$

$$\det\big(J(E^*)\big) = \left[A_1 - A_2 - \beta(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi) - \frac{\rho}{\alpha_1}(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)\right]I^* + [-A_3 + \mu + \rho][(\alpha_1+\mu)(\delta+\mu+\alpha_2-\psi)] + [A_4 - A_5].$$

Where :

$$A_1 = \alpha_1\delta\mu\beta + \alpha_1\delta\rho\beta + \alpha_1\alpha_2\mu\beta + \alpha_1\alpha_2\beta + \alpha_1\mu^2\beta + \alpha_1\mu\rho\beta + \mu^2\delta\beta + \mu\delta\beta\rho + \mu^3\beta + \mu^2\rho\beta + \mu^2\alpha_1\beta + \mu\alpha_2\rho\beta + \rho\beta\alpha_1\alpha_2, A_2 = \alpha_1\mu\psi\beta + \alpha_1\rho\psi\beta + \mu^2\rho\psi + \mu\psi\rho\beta, A_3 = \mu^2 + \frac{\rho}{\beta}\mu.$$

$$A_4 = \alpha_1\delta\mu^2 + \alpha_1\rho\delta\mu + \alpha_1\alpha_2\mu^2 + \alpha_1\alpha_2\rho\mu + \alpha_1\mu^3 + \alpha_1\mu^2 + \delta\mu^3 + \delta\mu^2\rho + \mu^4 + \rho\mu^3 + \alpha_2\mu^3 + \alpha_2\mu^2\rho$$

$A_5 = \alpha_1\psi\mu^2 + \alpha_1\psi\rho\mu + \mu^2\psi\rho$

And if $c_1 = \alpha_1 + \mu\,, c_2 = \delta + \mu + \alpha_2, c_3 = (\mu + \rho)$ then $\det\big(J(E^*)\big)$ becomes.

$$\det\big(J(E^*)\big) = \left[A_1 + \beta\psi c_1 + \frac{\rho\psi}{\alpha_1}c_1 - (A_2 + \beta c_1c_2 + \frac{\rho}{\alpha_1}c_1c_2)\right][\frac{\alpha_1\beta\Lambda c_3 + \mu\psi c_1c_3 - \mu\psi c_2c_3}{\beta c_1c_2c_3 - (\beta\psi c_1c_3 + \alpha_1\alpha_2\rho\beta)}]$$
$$+c_1\psi A_3 + c_1c_2c_3 + A_4 - (c_1c_2A_3 + c_1c_3\psi + A_5).$$

If $f_1 = A_1 + \beta\psi c_1 + \frac{\rho\psi}{\alpha_1}c_1, f_2 = A_2 + \beta c_1c_2 + \frac{\rho}{\alpha_1}c_1c_2, f_3 = \alpha_1\beta\Lambda c_3 + \mu\psi c_1c_3, f_4 = \mu\psi c_2c_3$

$f_5 = \dfrac{\alpha_1\beta\Lambda c_3 + \mu\psi c_1c_3 - \mu\psi c_2c_3}{\beta c_1c_2c_3 - (\beta\psi c_1c_3 + \alpha_1\alpha_2\rho\beta)}\,, f_6 = \beta\psi c_1c_3 + \alpha_1\alpha_2\rho\beta\,, f_7 = c_1\psi A_3 + c_1c_2c_3 + A_4$

And $f_8 = c_1c_2A_3 + c_1c_3\psi + A_5$

$$\det\big(J(E^*)\big) = -\left[\frac{f_1f_4 + f_2f_3 + f_7f_5 + f_5f_8 - (f_1f_3 + f_2f_4 + f_5f_7 + f_6f_8)}{f_5 - f_6}\right] < 0$$

If $f_1f_4 + f_2f_3 + f_7f_5 + f_5f_8 > f_1f_3 + f_2f_4 + f_5f_7 + f_6f_8$ and $f_5 > f_6$

Next, we calculate the determinant of $J^{[2]}(E^*)$ in (3.10) and obtained

$$= \begin{bmatrix} -(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi) & 0 & 0 \\ \alpha_2 & -(\alpha_1 + 2\mu + \rho) & \beta S^* \\ 0 & \alpha_1 & -(\delta + 2\mu + \alpha_2 + \rho - \psi) \end{bmatrix}$$

$$\det[J^{[2]}(E^*)] = (\beta I^* + 2\mu + \alpha_1)(\beta I^* + 2\mu + \delta + \alpha_2 - \psi)(\beta I^* + 2\mu + \rho)(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi)$$
$$(\alpha_1 + 2\mu + \rho)(\delta + \alpha_2 + 2\mu + \rho - \psi) - (\beta I^* + 2\mu + \alpha_1)(\beta I^* + 2\mu + \delta + \alpha_2 - \psi)(\beta I^* + 2\mu + \rho)$$
$$(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi)(\alpha_1)(\beta I^*) + \rho\alpha_1\alpha_2(\beta I^* + 2\mu + \alpha_1)(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi)(\beta I^*)$$
$$-\alpha_1\beta S^*(\beta I^* + 2\mu + \rho)(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi)(\alpha_1 + 2\mu + \rho)(2\mu + \alpha_2 + \delta - \psi)$$
$$+\alpha_1\beta S^*(\beta I^* + 2\mu + \rho)(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi)(\alpha_1 + 2\mu + \rho)(\alpha_1\beta S^*)$$
$$-\alpha_1\beta S^*(\beta I^* + 2\mu + \rho)(\alpha_1 + 2\mu + \alpha_2 + \delta - \psi)(\alpha_1\beta S^*\beta I^*)$$
$$-\alpha_1\beta S^*(\beta I^* + 2\mu + \rho)(\beta I^*)(\rho + 2\mu + \alpha_2 + \delta - \psi)(\alpha_1 + 2\mu + \rho)$$
$$+\alpha_1\alpha_1\beta S^*\beta S^*\beta I^*(\beta I^* + 2\mu + \rho)(\beta I^*) - \alpha_1\alpha_1\beta S^*\beta S^*\beta I^*\beta I^*$$

If $\det[J^{[2]}(E^*)] < 0$, this completes the proof.

4 Application of the Model to Malaria Transmission

4.1 Model description and basic reproduction number

Here we apply the SEIR model to malaria disease. The model is formulated for both human population as well as mosquito population at time t, for human we divide our population into four classes Susceptible S_H , Exposed E_H, Infectious I_H, and Recovery Human R_H. And the population of the mosquitoes is divided into three classes, namely susceptible S_V, Exposed E_V, and Infectious I_v.

The interaction between the human and mosquitoes is shown in the schematic diagram in Fig. 2, below:

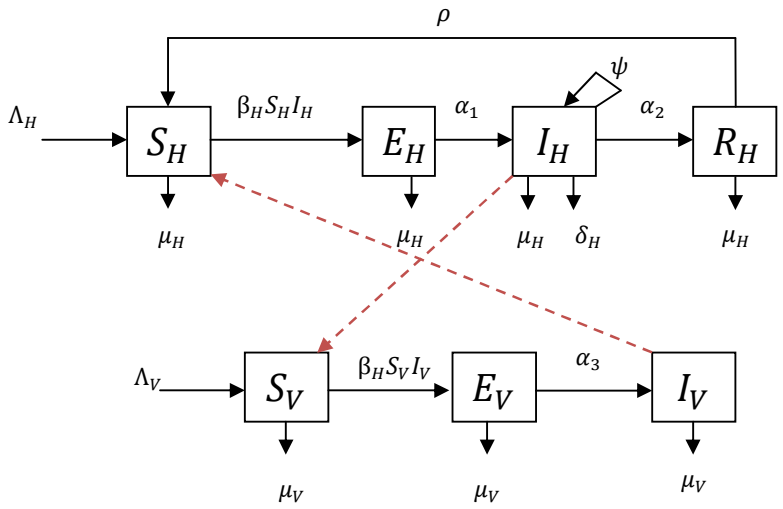

Fig. 2. Schematic diagram of malaria transmission between human and mosquito

Table 2. list of the parameters

Parameter	Description
Λ_H	Recruitment rate of humans
Λ_v	Recruitment rate of mosquitoes
α_1	Developing rate of exposed (humans) becoming infectious
α_2	Recover rate of humans.(removal rate)
μ_H	Natural death rate of humans
δ	Induce death rate of humans
α_3	Developing rate of exposed (mosquitoes) becoming infectious
μ_V	Natural death rate of mosquitoes
q_H	Probability of transmission of infection from an infectious mosquitoes to a susceptible humans
q_V	Probability of transmission of infection from an infectious humans to a susceptible mosquitoes
η_V	Mosquitoes biting rate
β_H	Infection rate $q_H \times \eta_V$ of humans
β_V	Infection rate $q_v \times \eta_V$ mosquitoes
ρ	Loss of immunity for humans
ψ	Rate of the newborn's birth with infection humans

The model equations are given by:

$$\begin{cases} \frac{dS_H}{dt} = \Lambda_H - \beta_H S_H I_H - \mu_H S_H + \rho R_H \\ \frac{dE_H}{dt} = \beta_H S_H I_H - (\alpha_1 + \mu_H) E_H \\ \frac{dI_H}{dt} = \alpha_1 E_H - (\alpha_2 + \mu_H + \delta) I_H + \psi I_H \\ \frac{dR_H}{dt} = \alpha_2 I_H - (\mu_H + \rho) R_H \\ \frac{dS_V}{dt} = \Lambda_V - \beta_V S_V I_V - \mu_V S_V \\ \frac{dE_V}{dt} = \beta_V S_V I_V - (\alpha_3 + \mu_V) E_V \\ \frac{dI_V}{dt} = \alpha_3 E_V - \mu_V I_V \end{cases} \quad (4.1)$$

Where: $S_H(t) = S_H(0)$, $E_H(t) = E_H(0), I_H(t) = I_H(0), R_H(t) = R_H(0), S_V(t) = S_V(0),$

$$E_V(t) = E_V(0), I_V(t) = I_V(0)$$

We also consider the following equations:

$$N_H(t) = S_H(t) + E_H(t) + I_H(t) + R_H(t) \tag{4.2}$$

Then the derivative of $N_H(t)$ with respect to t is given by:

$$\frac{dN_H}{dt} = \frac{dS_H}{dt} + \frac{dE_H}{dt} + \frac{dI_H}{dt} + \frac{dR_H}{dt}$$
$$\frac{dN_H}{dt} = \Lambda_{\mathrm{H}} - \mu_H N_H - (\delta_H - \psi) I_H$$

$$\frac{dN_H}{dt} \leq \Lambda_{\mathrm{H}} - \mu_H N_H$$
$$\lim_{t\to\infty} N_H(t) \leq \frac{\Lambda_{\mathrm{H}}}{\mu_H}.$$

Thus the feasible region of the system (4.1) for the human is given by

$$\Gamma^{*} = \{(S_H, E_H, I_H, R_H): S_H + E_H + I_H + R_H \leq \frac{\Lambda_{\mathrm{H}}}{\mu_H}, S_H > 0, E_H \geq 0, I_H \geq 0, R_H \geq 0\}$$

is positively invariant and

$$N_V(t) = S_V(t) + E_V(t) + I_V(t) \tag{4.3}$$

Then the derivative of $N_V(t)$ with respect to t is given by:

$$\frac{dN_V}{dt} = \frac{dS_V}{dt} + \frac{dE_V}{dt} + \frac{dI_V}{dt}$$
$$\frac{dN_V}{dt} \leq \Lambda_{\mathrm{V}} - \mu_V N_V$$
$$\lim_{t\to\infty} N_V(t) \leq \frac{\Lambda_{\mathrm{V}}}{\mu_V}.$$

Thus the feasible region of the system (4.1) for the mosquito is given by

$\Gamma^{**} = \{(S_V, E_V, I_V): S_V + E_V + I_V \leq \frac{\Lambda_{\mathrm{V}}}{\mu_V}, S_V > 0, E_V \geq 0, I_V \geq 0\}$ is positively invariant.

Then the system of ODE's (4.1) has a positive region defined as:

$\Gamma = \{(S_H, E_H, I_H, R_H, S_V, E_V, I_V): S_H(t) + E_H(t) + I_H(t) + R_H(t), S_V(t) + E_V(t) + I_V(t)$
$S_H > 0, E_H \geq 0, I_H \geq 0, R_H \geq 0, S_V > 0, E_V \geq 0, I_V \geq 0\}$, is positively invariant set.

Then it is easy to see that the system (4.1) has the disease-free equilibrium

$E_{0HV} = (\frac{\Lambda_{\mathrm{H}}}{\mu_H}, 0, 0, 0, \frac{\Lambda_{\mathrm{V}}}{\mu_V}, 0, 0)$. The basic reproduction number R_0 for the human and mosquito will be found by using the next generation matrix found in [12,17] as $R_1 = \sqrt{R_{0H} R_{0V}}$,then the R_0 the R_0 for the model (4.1) is calculated and defined by:

$$R_1 = \sqrt{\frac{\alpha_1 \alpha_2 \beta_V \beta_H \Lambda_{\mathrm{V}} \Lambda_{\mathrm{H}}}{\mu_H \mu_V^2 (\mu_V + \alpha_3)(\mu_H + \alpha_1)(\mu_H + \alpha_1 + \delta - \psi)}} \tag{4.4}$$

Theorem3: The disease-free equilibrium $E_{0HV}\left(\frac{\Lambda_H}{\mu_H}, 0, 0,0, \frac{\Lambda_V}{\mu_V}, 0,0\right)$ of the system (4.1) is asymptotically stable if $R_1 < 1$ and unstable if $R_1 > 1$.

4.2 Model analysis

4.2.1 Disease-free equilibrium

In this section, we investigate the local geometrical properties of the disease-free equilibrium $E_{0HV}\left(\frac{\Lambda_H}{\mu_H}, 0, 0,0, \frac{\Lambda_V}{\mu_V}, 0,0\right)$ by considering the linearized system of ODE's (4.1), taking the Jacobian matrix and obtained

$$J(S_H, E_H, I_H, R_H, S_V, E_V, I_V) = \begin{bmatrix} J_0 & J_2 \\ J_1 & J_3 \end{bmatrix} \tag{4.5}$$

Where:

$$J_0 = \begin{bmatrix} -\beta_H I_H - \mu_H & 0 & -\beta_H S_H \\ \beta_H I_H & -(\alpha_1 + \mu_H) & \beta_H S_H \\ 0 & \alpha_1 & -(\alpha_2 + \mu_H + \delta) + \psi \end{bmatrix}, J_1 = \begin{bmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, J_2 = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$J_3 = \begin{bmatrix} -(\mu_H + \rho) & 0 & 0 & 0 \\ 0 & -\beta_V I_V - \mu_V & 0 & -\beta_V S_V \\ 0 & \beta_H I_V & -(\alpha_3 + \mu_V) & \beta_V S_V \\ 0 & 0 & \alpha_3 & -\mu_V \end{bmatrix}$$

The local stability of the equilibrium may be determined from the Jacobian matrix (4.5). This implies that the Jacobian matrix for the disease -free equilibrium is given by

$$J(E_{0HV}) = \begin{bmatrix} J_0 & J_2 \\ J_1 & J_3 \end{bmatrix} \tag{4.6}$$

Where:

$$J_0 = \begin{bmatrix} -\mu_H & 0 & -\beta_H \frac{\Lambda_H}{\mu_H} \\ 0 & -(\alpha_1 + \mu_H) & \beta_H \frac{\Lambda_H}{\mu_H} \\ 0 & \alpha_1 & -(\alpha_2 + \mu_H + \delta) + \psi \end{bmatrix}, J_1 = \begin{bmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } J_3 = \begin{bmatrix} -(\mu_H + \rho) & 0 & 0 & 0 \\ 0 & -\mu_V & 0 & -\beta_V \frac{\Lambda_V}{\mu_V} \\ 0 & 0 & -(\alpha_3 + \mu_V) & \beta_V \frac{\Lambda_V}{\mu_V} \\ 0 & 0 & \alpha_3 & -\mu_V \end{bmatrix}$$

The determinant of (4.6) is given by:

$$J(E_{0HV} - \lambda I) = \begin{vmatrix} J_0 & J_2 \\ J_1 & J_3 \end{vmatrix} = 0 \tag{4.7}$$

Where: $J_0 = \begin{bmatrix} -\mu_H - \lambda & 0 & -\beta_H \frac{\Lambda_H}{\mu_H} \\ 0 & -(\alpha_1 + \mu_H) - \lambda & \beta_H \frac{\Lambda_H}{\mu_H} \\ 0 & \alpha_1 & -(\alpha_2 + \mu_H + \delta) + \psi - \lambda \end{bmatrix}$, $J_1 = \begin{bmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$J_2 = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $J_3 = \begin{bmatrix} -(\mu_H + \rho) - \lambda & 0 & 0 & 0 \\ 0 & -\mu_V - \lambda & 0 & -\beta_V \frac{\Lambda_V}{\mu_V} \\ 0 & 0 & -(\alpha_3 + \mu_V) - \lambda & \beta_V \frac{\Lambda_V}{\mu_V} \\ 0 & 0 & \alpha_3 & -\mu_V - \lambda \end{bmatrix}$

The eigenvalues of the (4.7) are given by:

Clearly $\lambda_1 = -\mu_H, \lambda_2 = -(\mu_H + \rho), \lambda_3 = -\mu_V$ are negatives and

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0 \tag{4.8}$$

By Using the Routh-Hurwitz criterion [14,15], it can be seen that all the eigenvalues of the characteristic equation (4.8) have negative real part if and only if:

$$\begin{cases} c_1 > 0,\ c_2 > 0, c_3 > 0, c_4 > 0, c_1c_2 - c_3 > 0, c_1c_2c_3 - {c_3}^2 - {c_1}^2c_4 > 0, \\ c_1c_2c_3c_4 - c_0{c_3}^2c_4 > 0 \end{cases} \tag{4.9}$$

Where: $c_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\mu_H + 2\mu_V + \delta - \psi$

$$c_2 = \alpha_1\alpha_2 + 2\alpha_1\mu_H + \delta\alpha_1 + \alpha_2\mu_H + {\mu_H}^2 + {\mu_V}^2 + \delta\mu_H + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\mu_V + 4\mu_H\mu_V + \alpha_2\mu_V + 2\alpha_3\mu_H + \delta\alpha_3 + 2\delta\mu_V + 2\alpha_3\mu_V - \alpha_1\psi - \mu_H\psi - \alpha_3\psi - 2\psi\mu_V - \alpha_1\frac{\beta_H\Lambda_H}{\mu_H} - \alpha_3\frac{\beta_V\Lambda_V}{\mu_V}$$

$$c_3 = \alpha_1{\alpha_2}^2 + 2\alpha_1\alpha_2\mu_V + 4\alpha_1\mu_H\mu_V + \alpha_1\alpha_3\mu_H + 2\alpha_1\alpha_3\mu_V + 2\mu_V{}^2 + \alpha_1\alpha_2\mu_H + 2{\mu_H}^2\mu_V + \alpha_3{\mu_H}^2 + \alpha_3\mu_H\mu_V + 2\mu_H{\mu_V}^2 + \delta\alpha_1\alpha_2 + 2\delta\alpha_1\mu_V + 2\delta\mu_H\mu_V + \delta\alpha_3\mu_H + \delta\alpha_3\mu_V + \delta{\mu_V}^2 + \alpha_3\mu_H + \alpha_3\psi\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_1\alpha_2\psi + 2\alpha_1\psi\mu_V - 2\mu_H\mu_V\psi - 2\alpha_3\mu_H\psi - \alpha_3\mu_V\psi - {\mu_V}^2\psi - \delta\alpha_3\frac{\beta_V\Lambda_V}{\mu_V} - \delta\alpha_3\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_1\alpha_3\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_3\mu_H\frac{\beta_V\Lambda_V}{\mu_V}.$$

$$c_4 = \alpha_1\alpha_2\alpha_3\mu_V + \alpha_1\alpha_2{\mu_V}^2 + \alpha_1\alpha_3\mu_H\mu_V + \alpha_1\mu_H{\mu_V}^2 + \alpha_1\alpha_3\delta\mu_V + \alpha_1\delta{\mu_V}^2 - \alpha_1\alpha_3\psi\mu_V - \alpha_1\psi{\mu_V}^2 + \alpha_2\alpha_3\mu_H\mu_V + \alpha_2\mu_H{\mu_V}^2 + \alpha_3{\mu_H}^2\mu_V + \mu_H{\mu_V}^2 + \delta\alpha_3\mu_H\mu_V + \delta\mu_H{\mu_V}^2 - \alpha_3\mu_H\mu_V\psi - \psi\mu_H{\mu_V}^2 - \alpha_1\alpha_2\alpha_3\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_1\alpha_3\mu_H\frac{\beta_V\Lambda_V}{\mu_V} - \delta\alpha_1\alpha_3\frac{\beta_V\Lambda_V}{\mu_V} + \alpha_1\psi\alpha_3\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_2\alpha_3\mu_H\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_2\alpha_3\mu_H\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_3{\mu_H}^2\frac{\beta_V\Lambda_V}{\mu_V} - \delta\alpha_3\mu_H\frac{\beta_V\Lambda_V}{\mu_V} + \alpha_3\psi\mu_H\frac{\beta_V\Lambda_V}{\mu_V} - \alpha_1\mu_V\frac{\beta_H\Lambda_H}{\mu_H} - \alpha_1\alpha_3\mu_V\frac{\beta_H\Lambda_H}{\mu_H} - \alpha_1{\mu_V}^2\frac{\beta_H\Lambda_H}{\mu_H} + \alpha_1\alpha_3\mu_V\frac{\beta_H\Lambda_H}{\mu_H}\frac{\beta_V\Lambda_V}{\mu_V}$$

It can be seen that all the eigenvalues have negative real parts and therefore the disease-free equilibrium is Locally Asymptotically Stable.

4.2.2 Endemic equilibrium

In this section, we consider a situation in which all the steady states coexist in the equilibrium. We denote $E^* = ({S_H}^*, {E_H}^*, {I_H}^*, {R_H}^*, {S_V}^*, {E_V}^*, {I_V}^*)$ as the endemic equilibrium of the system (4.1). We also obtain:

$${S_H}^* = \frac{(\alpha_1+\mu_H)(\delta+\mu_H+\alpha_2-\psi)}{\alpha_1\beta_H}\ ,\ {E_H}^* = \frac{(\mu_H+\rho)(\delta+\mu_H+\alpha_2-\psi)[\mu_H(\alpha_1+\mu_H)(\delta+\mu_H+\alpha_2-\psi)-\alpha_1\beta_H\Lambda_H}{\alpha_1[\beta_H(\mu_H+\rho)(\delta+\mu_H+\alpha_2-\psi)+\alpha_1\alpha_2\beta_H\rho]}\ ,$$

$$I_H{}^* = \frac{(\mu_H+\rho)[\mu_H(\alpha_1+\mu_H)(\delta+\mu_H+\alpha_2-\psi)-\alpha_1\beta_H\Lambda_H}{\beta_H(\mu_H+\rho)(\alpha_1+\mu_H)(\delta+\mu_H+\alpha_2-\psi)+\alpha_1\alpha_2\beta_H\rho},\ R_H{}^{**} = \frac{\alpha_1[\mu_H(\alpha_1+\mu_H)(\delta+\mu_H+\alpha_2-\psi)-\alpha_1\beta_H\Lambda_H]}{\beta_H(\mu_H+\rho)(\alpha_1+\mu_H)(\delta+\mu_H+\alpha_2-\psi)+\alpha_1\alpha_2\beta_H\rho]}$$
$$S_V{}^* = \frac{\mu_V(\alpha_3+\mu_V)}{\alpha_3\beta_V}, E_V{}^* = \frac{\alpha_3\beta_V\Lambda_V-\mu_V{}^2(\alpha_3+\mu_V)}{\alpha_3\beta_V(\alpha_3+\mu_V)}\ and\ I_V{}^* = \frac{\alpha_3\beta_V\Lambda_V-\mu_V{}^2(\alpha_3+\mu_V)}{\beta_V\mu_V(\alpha_3+\mu_V)}$$

from system of ODE's (1) and linearized the same system to obtained:

$$J(S_H{}^*, E_H{}^*, I_H{}^*, R_H{}^*, S_V{}^*, E_V{}^*, I_V{}^*) = \begin{bmatrix} J_0 & J_2 \\ J_1 & J_3 \end{bmatrix} \tag{4.10}$$

Where:

$$J_0 = \begin{bmatrix} -\beta_H I_H{}^* - \mu_H & 0 & -\beta_H S_H{}^* \\ \beta_H I_H{}^* & -(\alpha_1+\mu_H) & \beta_H S_H{}^* \\ 0 & \alpha_1 & -(\alpha_2+\mu_H+\delta_H)+\psi \end{bmatrix}, J_1 = \begin{bmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$J_2 = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } J_3 = \begin{bmatrix} -(\mu_H+\rho) & 0 & 0 & 0 \\ 0 & -\beta_V I_V{}^* - \mu_V & 0 & -\beta_V S_V{}^* \\ 0 & \beta_V I_V{}^* & -(\alpha_3+\mu_V) & \beta_V S_V{}^* \\ 0 & 0 & \alpha_3 & -\mu_V \end{bmatrix}$$

Therefore, the characteristic equation of (4.10) is given by

$$\lambda^7 + d_1\lambda^6 + d_2\lambda^5 + d_3\lambda^4 + d_4\lambda^3 + d_5\lambda^2 + d_6\lambda + d_7 = 0 \tag{4.11}$$

Where

$$d_1 = -(a_{11}+a_{22}+a_{33}+a_{44}+a_{55}+a_{66}+a_{77})$$

$$d_2 = a_{11}a_{22}+a_{11}a_{33}+a_{11}a_{44}+a_{11}a_{55}+a_{11}a_{66}+a_{11}a_{77}+a_{22}a_{33}+a_{22}a_{44}+a_{22}a_{55}+ a_{22}a_{66}+a_{22}a_{77}+a_{33}a_{44}+a_{33}a_{55}+a_{33}a_{66}+a_{33}a_{77}+a_{44}a_{55}+a_{44}a_{66}+a_{44}a_{77}+a_{55}a_{66} +a_{55}a_{77}+a_{66}a_{77}$$

$$d_3 = -[a_{11}a_{22}a_{33}+a_{11}a_{22}a_{44}+a_{11}a_{22}a_{55}+a_{11}a_{22}a_{66}+a_{11}a_{22}a_{77}+a_{11}a_{33}a_{44}+a_{11}a_{33}a_{55} +a_{11}a_{33}a_{66}+a_{11}a_{33}a_{77}+a_{11}a_{44}a_{55}+a_{11}a_{44}a_{66}+a_{11}a_{44}a_{77}+a_{11}a_{55}a_{66}+a_{11}a_{55}a_{66}+ a_{11}a_{55}a_{77}+a_{11}a_{66}a_{77}+a_{22}a_{33}a_{44}+a_{22}a_{33}a_{55}+a_{22}a_{33}a_{77}+a_{22}a_{44}a_{55}+a_{22}a_{44}a_{66}+ a_{22}a_{44}a_{77}+a_{22}a_{55}a_{66}+a_{22}a_{55}a_{77}+a_{22}a_{66}a_{77}+a_{33}a_{44}a_{55}+a_{33}a_{44}a_{66}+a_{33}a_{44}a_{77}+ a_{33}a_{55}a_{66}+a_{33}a_{55}a_{77}+a_{33}a_{66}a_{77}+a_{44}a_{55}a_{66}+a_{44}a_{55}a_{77}+a_{44}a_{66}a_{77}+a_{55}a_{66}a_{77}].$$

$$d_4 = a_{11}a_{22}a_{33}a_{44}+a_{11}a_{22}a_{33}a_{66}+a_{11}a_{22}a_{33}a_{77}+a_{11}a_{22}a_{44}a_{55}+a_{11}a_{22}a_{44}a_{66}+ a_{11}a_{22}a_{44}a_{77}+a_{11}a_{22}a_{55}a_{66}+a_{11}a_{22}a_{55}a_{77}+a_{11}a_{22}a_{66}a_{77}+a_{11}a_{33}a_{44}a_{55}+ a_{11}a_{33}a_{44}a_{66}+a_{11}a_{33}a_{44}a_{77}+a_{11}a_{33}a_{55}a_{66}+a_{11}a_{33}a_{55}a_{77}+a_{11}a_{33}a_{66}a_{77}+ a_{11}a_{44}a_{55}a_{66}+a_{11}a_{44}a_{55}a_{77}+a_{11}a_{44}a_{66}a_{77}+a_{11}a_{55}a_{66}a_{77}+a_{22}a_{33}a_{44}a_{55}+ a_{22}a_{33}a_{44}a_{66}+a_{22}a_{33}a_{44}a_{77}+a_{22}a_{33}a_{55}a_{66}+a_{22}a_{33}a_{55}a_{77}+a_{22}a_{33}a_{66}a_{77}+ -a_{22}a_{33}a_{66}+a_{22}a_{44}a_{55}a_{66}+a_{22}a_{44}a_{55}a_{77}+a_{22}a_{44}a_{66}a_{77}+a_{22}a_{55}a_{66}a_{77}+ a_{33}a_{44}a_{55}a_{66}+a_{33}a_{55}a_{66}a_{77}+a_{33}a_{44}a_{55}a_{77}+a_{33}a_{44}a_{66}a_{77}+a_{44}a_{55}a_{66}a_{77}- a_{14}a_{21}a_{32}a_{43}.$$

$$d_5 = -[a_{11}a_{22}a_{33}a_{44}a_{55} + a_{11}a_{22}a_{33}a_{44}a_{66} + a_{11}a_{22}a_{33}a_{44}a_{77} + a_{11}a_{22}a_{33}a_{55}a_{66} + a_{11}a_{22}a_{33}a_{55}a_{77} - a_{11}a_{22}a_{33}a_{55} + a_{11}a_{22}a_{33}a_{66}a_{77} + a_{11}a_{22}a_{44}a_{55}a_{77} + a_{11}a_{22}a_{44}a_{66}a_{77} + a_{11}a_{22}a_{55}a_{66}a_{77} + a_{11}a_{33}a_{44}a_{55}a_{66} + a_{11}a_{33}a_{44}a_{66}a_{77} + a_{11}a_{33}a_{55}a_{66}a_{77} + a_{11}a_{55}a_{44}a_{55}a_{66}a_{77} + a_{22}a_{33}a_{44}a_{55}a_{77} + a_{22}a_{33}a_{44}a_{66}a_{77} + a_{22}a_{33}a_{55}a_{66}a_{77} + a_{22}a_{44}a_{55}a_{66}a_{77} + a_{33}a_{44}a_{55}a_{66}a_{77} - a_{14}a_{21}a_{32}a_{43}a_{77} - a_{14}a_{21}a_{32}a_{43}a_{55} - a_{14}a_{21}a_{32}a_{43}a_{66}$$

$$d_6 = a_{11}a_{22}a_{33}a_{44}a_{55}a_{66} + a_{11}a_{22}a_{33}a_{44}a_{55}a_{77} + a_{11}a_{22}a_{33}a_{44}a_{66}a_{77} + a_{11}a_{22}a_{33}a_{55}a_{66}a_{77} + a_{11}a_{22}a_{44}a_{55}a_{66}a_{77} + a_{11}a_{33}a_{44}a_{66}a_{77} + a_{22}a_{33}a_{44}a_{55}a_{66}a_{77} - a_{14}a_{21}a_{32}a_{43}a_{55}a_{77} - -a_{14}a_{21}a_{32}a_{43}a_{66}a_{77} - a_{14}a_{21}a_{32}a_{43}a_{55}a_{66}$$

$$d_7 = -a_{11}a_{22}a_{33}a_{44}a_{55}a_{66}a_{77} + a_{14}a_{21}a_{32}a_{43}a_{55}a_{66}a_{77} + a_{14}a_{21}a_{32}a_{43}a_{67}a_{76} + a_{14}a_{21}a_{32}a_{43}a_{57}a_{67}a_{76}$$

And

$$a_{11} = -(\beta_H {I_H}^* + \mu_H), a_{13} = -\beta_H {S_H}^*, a_{14} = \rho\,, a_{21} = \beta_H {I_H}^*, a_{22} = -(\alpha_1 + \mu_H), a_{23} = \beta_H {S_H}^*$$

$$a_{32} = \alpha_1, a_{33} = -(\alpha_2 + \mu_H + \delta - \psi), a_{43} = \alpha_2, a_{44} = -(\mu_H + \rho), a_{55} = -(\beta_V {I_V}^* + \mu_V)$$

$$a_{57} = -\beta_V {S_V}^*, a_{65} = \beta_V {I_V}^*, \qquad a_{66} = -(\alpha_3 + \mu_V), a_{67} = \beta_V {S_V}^*, a_{76} = \alpha_3, a_{77} = -\mu_V$$

By Using the Routh-Hurwitz criterion [14,15,18], it can be seen that all the eigenvalues of the characteristic equation (4.8) have negative real part if and only if: $d_i > 0$ $(i = 1,2,3,4,5,6,7)$.

$$D_1 = d_1 > 0, D_2 = \begin{vmatrix} d_1 & d_3 \\ 1 & d_2 \end{vmatrix} > 0, D_3 = \begin{vmatrix} d_1 & d_3 & d_5 \\ 1 & d_2 & d_4 \\ 0 & d_1 & d_3 \end{vmatrix} > 0, D_4 = \begin{vmatrix} d_1 & d_3 & d_5 & 0 \\ 1 & d_2 & d_4 & d_6 \\ 0 & d_1 & d_3 & d_5 \\ 0 & 1 & d_2 & d_4 \end{vmatrix}$$

$$D_5 = \begin{vmatrix} d_1 & d_3 & d_5 & d_7 & 0 \\ 1 & d_2 & d_4 & d_6 & 0 \\ 0 & d_1 & d_3 & d_5 & d_7 \\ 0 & 1 & d_2 & d_4 & d_6 \\ 0 & 0 & d_1 & d_3 & d_5 \end{vmatrix} > 0, D_6 = \begin{vmatrix} d_1 & d_3 & d_5 & d_7 & 0 & 0 \\ 1 & d_2 & d_4 & d_6 & 0 & 0 \\ 0 & d_1 & d_3 & d_5 & d_7 & 0 \\ 0 & 1 & d_2 & d_4 & d_6 & 0 \\ 0 & 0 & d_1 & d_3 & d_5 & d_7 \\ 0 & 0 & 1 & d_2 & d_4 & d_6 \end{vmatrix} \tag{4.12}$$

$$D_7 = \begin{vmatrix} d_1 & d_3 & d_5 & d_7 & 0 & 0 & 0 \\ 1 & d_2 & d_4 & d_6 & 0 & 0 & 0 \\ 0 & d_1 & d_3 & d_5 & d_7 & 0 & 0 \\ 0 & 1 & d_2 & d_4 & d_6 & 0 & 0 \\ 0 & 0 & d_1 & d_3 & d_5 & d_7 & 0 \\ 0 & 0 & 1 & d_2 & d_4 & d_6 & 0 \\ 0 & 0 & 0 & d_1 & d_3 & d_5 & d_7 \end{vmatrix} > 0$$

Therefore the system (4.1) shows local asymptotic stability at E^* when $R_0 > 1$ which guarantees the existence of E^* and conditions (4.12) are satisfied. So, we arrive to the following result.

Theorem 4: The endemic equilibrium E^* of the system (4.1) is locally asymptotically stable if $R_0 > 1$ and conditions (4.12) are satisfied.

5 Numerical Simulations

In this section, we use numerical simulations to show the dynamical behaviour of our results the parameters that would be used in this section are displayed in Table 3, Table 4 and under figures.

Table 3. Parameters values of model 1

Parameter	Description	Value	Source
Λ	Recruitment rate of susceptible.	1.2	[11]
β	Infection rate(effective infection rate)	0.001	Assumed
α_1	Developing rate of exposed (humans)	0.1	Assumed
α_2	Recover rate of humans (removal rate)	0.0035	[12]
μ	Natural death rate	0.03	[11]
δ	induce death rate	0.089	[11]
ψ	Rate of the newborn's birth with infection humans	0.003	Assumed
ρ	Loss of immunity of humans	0.00017	[19]

Table 4. Parameters values of model 2

Parameter	Description	Value	Source
Λ_H	Recruitment rate of humans	1.2	Assumed
Λ_v	Recruitment rate of mosquito	0. 7	Assumed
α_1	Developing rate of exposed (humans) becoming infectious	0.05	[9]
α_2	Recover rate of humans (removal rate)	0.0035	[12]
μ_H	Natural death rate of humans	0.01146	Assumed
δ	Induce death rate of humans	0.068	Assumed
α_3	Developing rate of exposed (mosquitoes) becoming infectious	0. 083	[12]
μ_V	Natural death rate of mosquitoes	0.05	[20]
q_H	Probability of transmission of infection from an infectious mosquitoes to susceptible humans	0.022	[21]
q_V	Probability of transmission of infection from an infectious humans to a susceptible mosquitoes	0.24	[21]
η_V	Mosquitoes biting rate	0.46	[2]
β_H	Infection rate $q_H \times \eta_V$ of humans	0.00638	[21]
β_V	Infection rate $q_v \times \eta_V$ mosquitoes	0.0696	[21]
ψ	Rate of the newborn's birth with infection humans	0.003	Assumed
ρ	Loss of immunity for humans	0.00017	[19]

5.1 Sensitivity analysis of the basic reproductive numbers

We investigate the nature of the model by conducting sensitivity analysis of the reproductive numbers R_0 , R_1 for model (2.1) and (4.1) respectively.

(a) At the disease-free equilibrium for (2.1): $\Lambda = 1.2, \beta = 0.001, \alpha_1 = 0.1, \alpha_2 = 0.0035, \mu = 0.03, \delta = 0.089, \psi = 0.003, R_0 < 1$.

(i) If the value of β is increased to 0.02 or more and the values ofα_1, α_2, μ , Λ, δ and ψ maintains same then $R_0 > 1$.

(ii) If the value of Λ is increased to 10 or more and the values of α_1 , α_2 , μ , β , δ and ψ maintains same then $R_0 > 1$.

(b) At the endemic equilibrium for (4.1): $\Lambda_H = 1.2, \beta_H = 0.00638, \alpha_1 = 0.05,\ \alpha_2 = 0.0035, \mu_H = 0.01146, \delta = 0.068, \rho = 0.00017$, $\psi = 0.003$, $\Lambda_V = 0.7, \beta_V = 0.0696$, $\alpha_3 = 0.083$, $\mu_V = 0.05$.

(i) If the values of β_H and β_v is decreased to 0.0000638 and 0.000696 respectively or more and the values of the others parameters maintains same then $R_0 < 1$.

(ii) If the values of β_H decreased from 0.638 to 0.0000638 or less and the values of the others parameters maintains same then $R_0 < 1$.

(iii) If the values of β_v decreased from 0.0696 to 0.0000696 or less and the values of the others parameters maintains same then $R_0 < 1$.

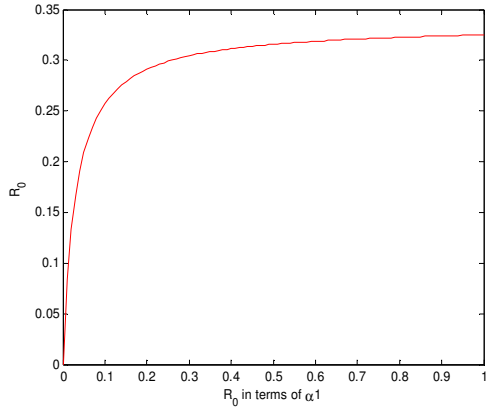

Fig. 3. The relationship between $R_0$and α_1.

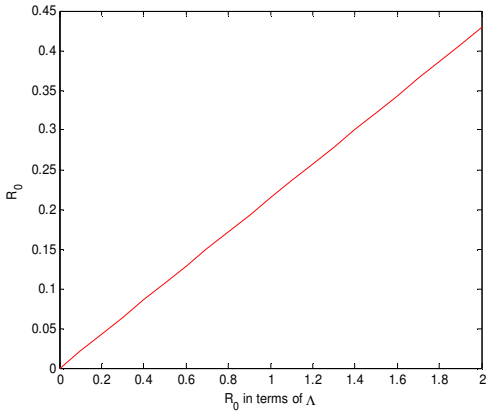

Fig. 4. The relationship between $R_0$and Λ

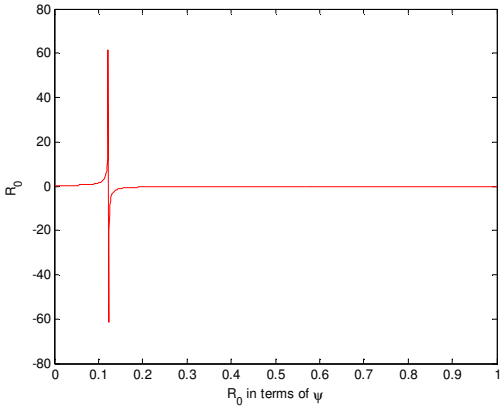

Fig. 5. The relationship between $R_0$and ψ.

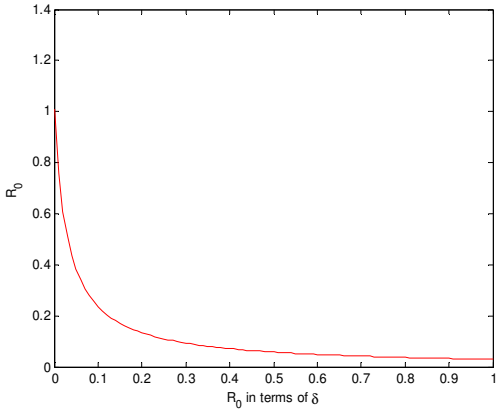

Fig. 6. The relationship between $R_0$and δ.

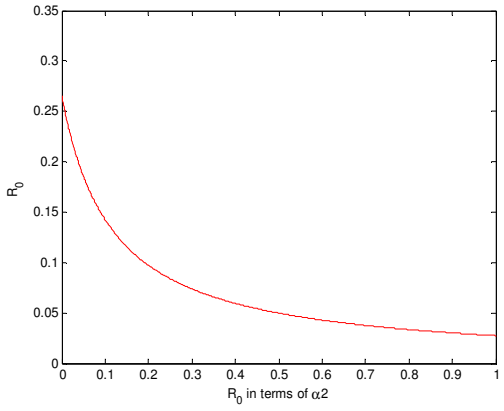

Fig. 7. The relationship between $R_0$and α_2.

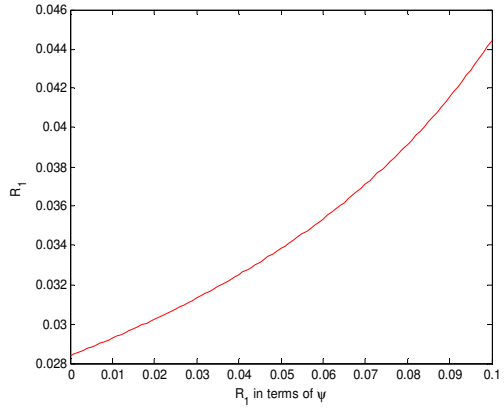

Fig. 8. The relationship between $R_1$and ψ.

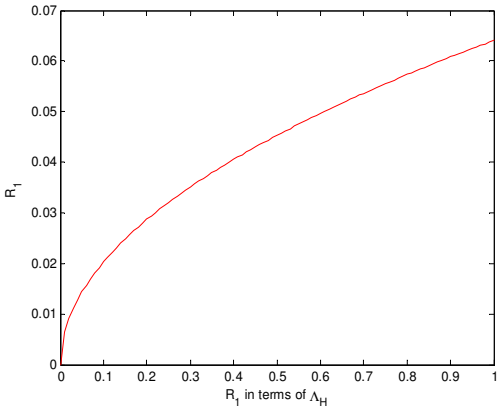

Fig. 9. The relationship between $R_1$and Λ_H.

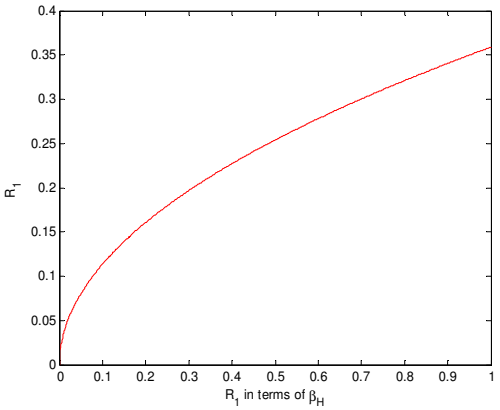

Fig. 10. The relationship between $R_1$and β_H.

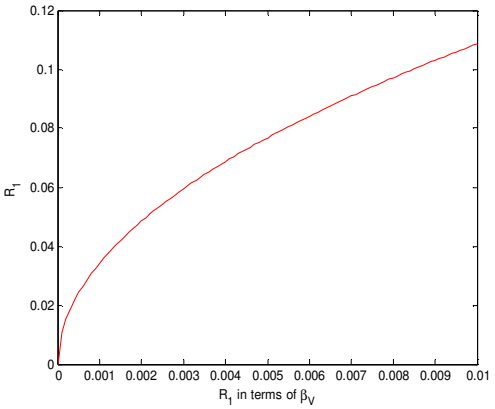

Fig. 11. The relationship between $R_1$and β_V.

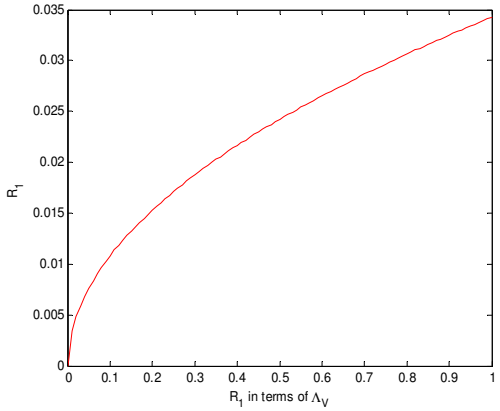

Fig. 12. The relationship between $R_0$and Λ_V.

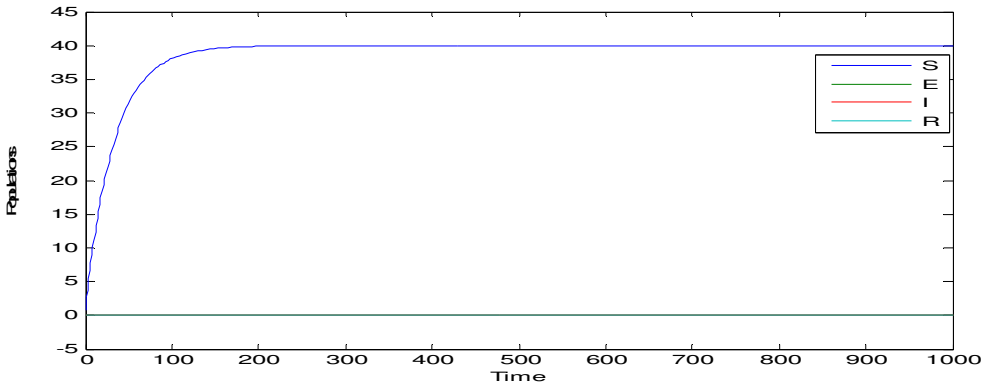

Fig. 13. Time responses of the state variables $S, E, I,$ and R with initial conditions $s(0)0.83, E(0) = 0.08, I(0) = 0.07, R(0) = 0.02$ Against the time and $R_0 = 0.2575$.Where the parameters: $\Lambda = 1.2, \beta = 0.001, \alpha_1 = 0.1, \alpha_2 = 0.0035, \mu = 0.03, \delta = 0.089, \rho = 0.00017, \psi = 0.003$

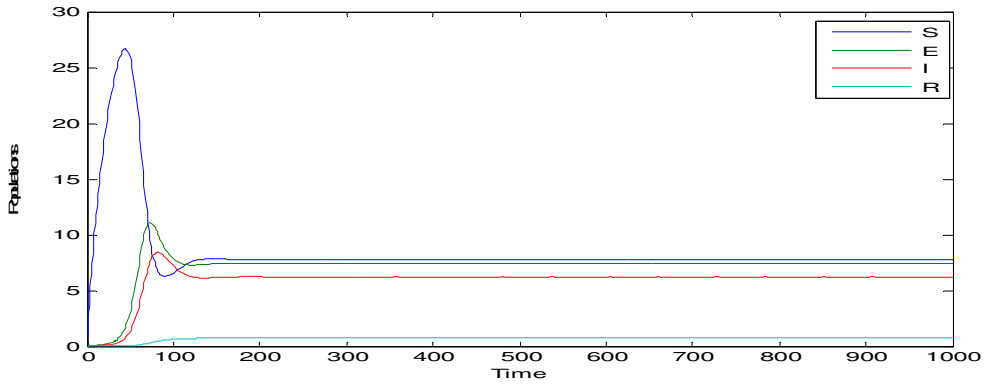

Fig. 14. Time responses of the state variables S, E, I, and R with initial conditions $s(0)0.83, E(0) = 0.08, I(0) = 0.07, R(0) = 0.02$ Against the time and $R_0 > 1$.Where the parameters: $\Lambda = 1.2, \beta = 0.02, \alpha_1 = 0.1, \alpha_2 = 0.0035, \mu = 0.03, \delta = 0.089, \rho = 0.00017, \psi = 0.003$

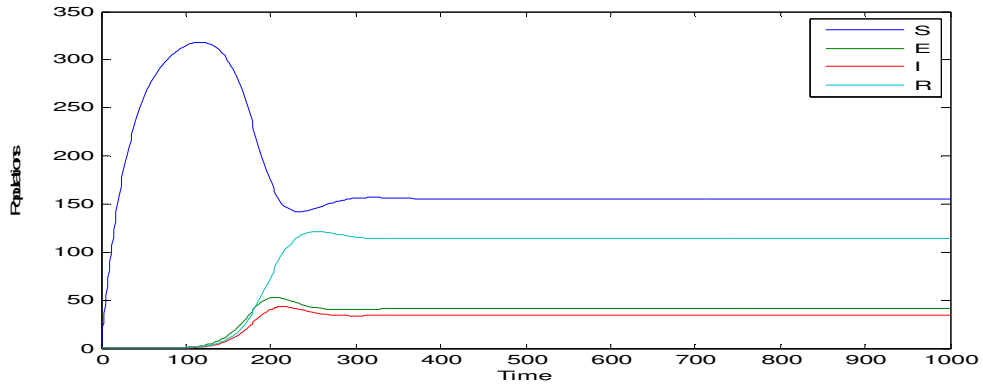

Fig. 15. Time responses of the state variables S, E, I, and R with initial conditions $s(0)0.83, E(0) = 0.08, I(0) = 0.07, R(0) = 0.02$ Against the time and $R_0 > 1$.Where the parameters: $\Lambda = 10, \beta = 0.001, \alpha_1 = 0.1, \alpha_2 = 0.0035, \mu = 0.03, \delta = 0.089, \rho = 0.00017, \psi = 0.003$

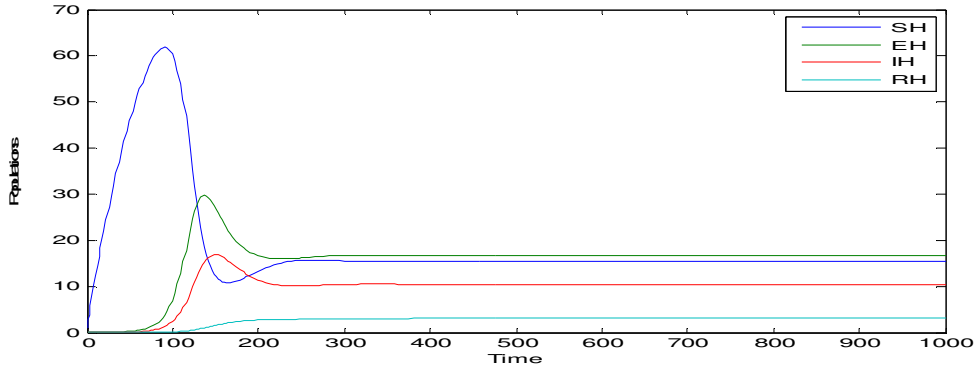

Fig. 16. Time responses of the state variables S_H, E_H, I_H, and R_H with initial conditions $s(0)0.83, E(0) = 0.08, I(0) = 0.07, R(0) = 0.02$ Against the time and $R_0 > 1$.Where the parameters: $\Lambda_H = 1.2, \beta_H = 0.00638, \alpha_1 = 0.05, \alpha_2 = 0.0035, \mu_H = 0.01146, \delta = 0.068, \rho = 0.00017, \psi = 0.003$.

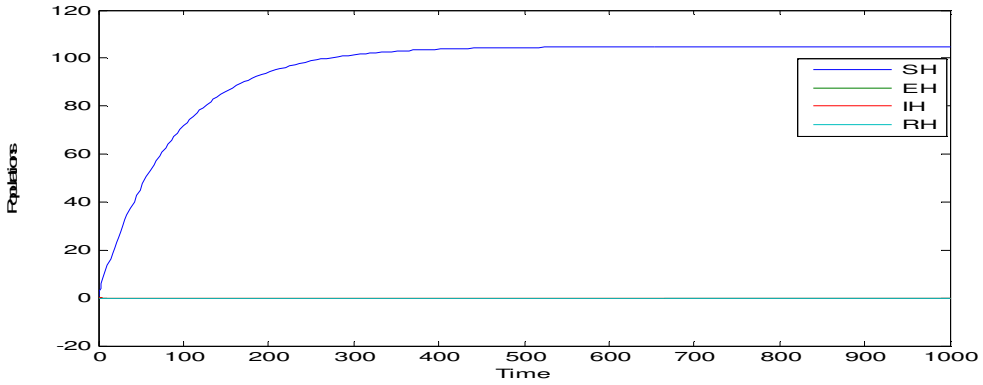

Fig. 17. Time responses of the state variables S_H, E_H, I_H, and R_H with initial conditions $s(0)0.83, E(0) = 0.08, I(0) = 0.07, R(0) = 0.02$ Against the time and $R_0 = 0.606$.Where The parameters: $\Lambda_H = 1.2, \beta_H = 0.000638, \alpha_1 = 0.001, \alpha_2 = 0.0035, \mu_H = 0.01146,$ $\delta =, 0.089, \rho = 0.00017, \psi = 0.003$

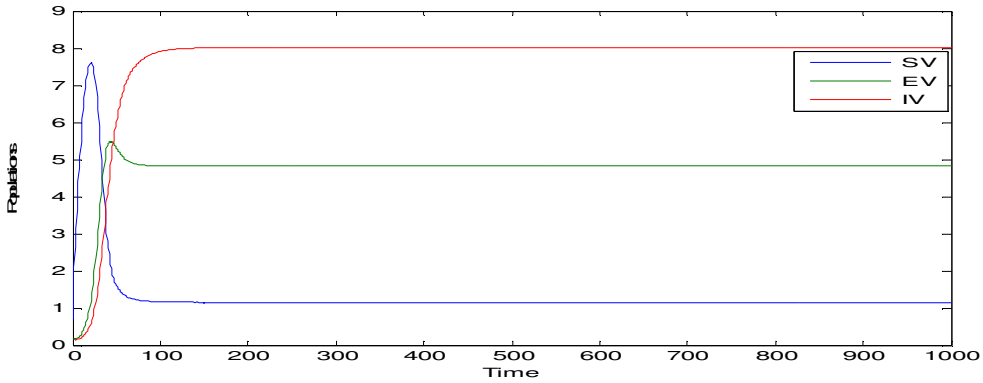

Fig. 18. Time responses of the state variables S_V, E_V, and I_V, with initial conditions $S_V(0) =, E_V(0) = 0.2, E_V(0) = 0.1$. Where the parameters: $\Lambda_V = 0.7, \beta_V = 0.0696, \alpha_3 = 0.083, \mu_V = 0.05$.

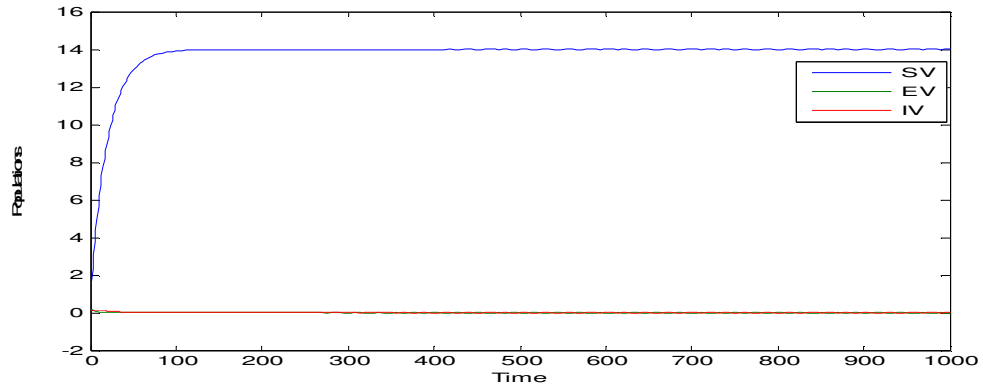

Fig. 19. Time responses of the state variables S_V, E_V, and I_V, with initial conditions $S_V(0) = 0.7$ $E_V(0) = 0.2, E_V(0) = 0.1$. Where the parameters: $\Lambda_V = 0.7, \beta_V = 0.000696, \alpha_3 = 0.083, \mu_V = 0.05$.

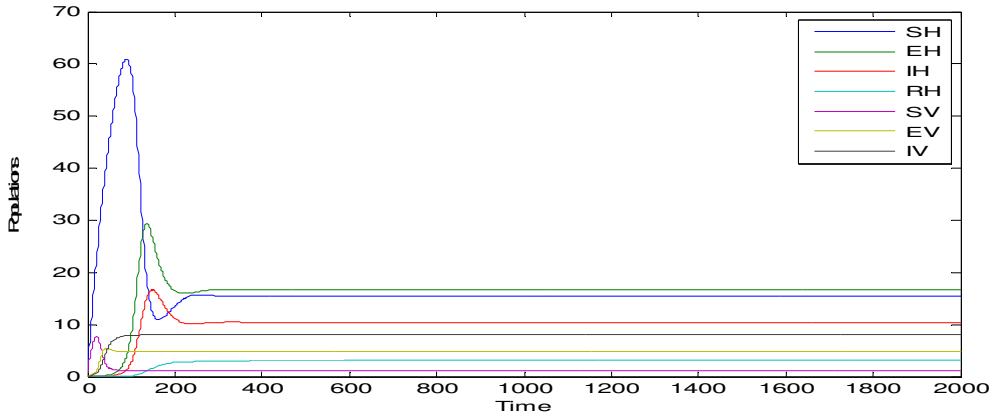

Fig. 20. Time responses of the state variables $S_H, E_H, I_H, R_H, S_V, E_V$, and I_V with initial conditions $s(0) = 0.83$, $E(0) = 0.08$, $I(0) = 0.07, R(0) = 0.02, S_V(0) = 0.7, E_V(0) = 0.2$, $E_V(0) = 0.1$. Against the time and $R_0 > 1$.Where the parameters: $\Lambda_H = 1.2, \beta_H = 0.00638, \alpha_1 = 0.05$, $\alpha_2 = 0.0035, \mu_H = 0.01146, \delta = 0.068, \rho = 0.00017, \psi = 0.003, \Lambda_V = 0.7, \beta_V = 0.0696$, $\alpha_3 = 0.083, \mu_V = 0.05$

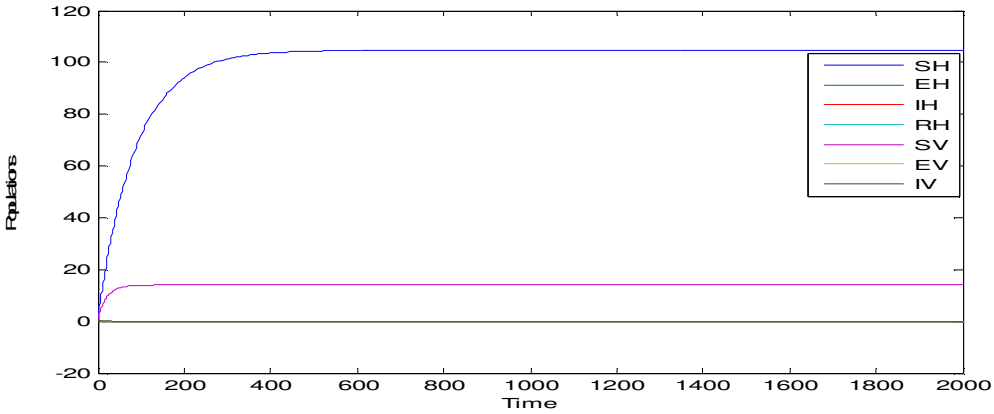

Fig. 21. Time responses of the state variables $S_H, E_H, I_H, R_H, S_V, E_V$, and I_V with initial conditions $s(0) = 0.83$, $E(0) = 0.08$, $I(0) = 0.07, R(0) = 0.02, S_V(0) = 0.7, E_V(0) = 0.2$, $E_V(0) = 0.1$. Against the time and $R_0 < 1$.Where the parameters: $\Lambda_H = 1.2, \beta_H = 0.000638, \alpha_1 = 0.05$, $\alpha_2 = 0.0035, \mu_H = 0.01146, \delta = 0.068, \rho = 0.00017, \psi = 0.003, \Lambda_V = 0.7, \beta_V = 0.000696$, $\alpha_3 = 0.083, \mu_V = 0.05$

6 Discussion of Results

In this paper, we studied the dynamics of an SEIR model and applied it to malaria transmission between human and mosquito, by including the newborn's birth with infection can enter infection compartment. We derived the basic reproduction number and discussed the existence and stability of Disease-Free Equilibrium (DFE) and Endemic Equilibrium (EE) of model (2.1) and applied them to model transmission between human and mosquitoes (4.1).

Our analysis shows that if the reproduction number is less than one then the (DFE) is locally asymptotically stable, this implies that only susceptible is present and the other populations approaches zero, and the disease dies out. And if the reproduction number is greater than one then (DFE) is unstable, this implies that all the

populations exist for (2.1) and (4.1) respectively. This situation has been verified numerically by simulations in Figs. (13,17,19,21) and (14,15,16,18,20) respectively , and Figs. (3,4,5,6,7,8),(9,10,11,12) show the graphs of $R_0 < 1$ and $R_1 < 1$ in terms of $\alpha_1, \alpha_2, \Lambda, \psi, \delta$ and $\beta_H, \beta_V, \Lambda_H, \Lambda_V, \psi$ respectively. Our sensitivity analysis shows that the most effective parameters are: β, Λ and β_H, β_V for model (2.1) and (4.1), the simulation result shows that both human and mosquitoes will exist and get infected. These results are helpful in predicting malaria transmission and how to find effective way of prevention and control in the model. Clearly, from the simulations the (DFE) is asymptotically stable whenever the reproduction number is less than one, the (EE) is unstable when the reproduction number is greater than one, of model (2.1) and (4.1) respectively. We notice that in order to reduce the basic reproduction number below one, we need to focused on reduction of contact between mosquitoes and human as well as the infection rate of human.

7 Conclusion

Our model shows that, Malaria cannot only be controlled by reducing the contact rate between the mosquito and humans, but also by reducing the infection rate between the human, and the use of active malaria drugs, insecticides, and, treated bed nets would reduce the mosquito population, and that keep the human population stable.

Acknowledgements

The authors thank Dr. Ebrahim AS-Shareef because the idea of this model came from his PhD Thesis [22].

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Jia Li. Modelling of transgenic mosquitoes and impact on malaria transmission. Journal of Biological Dynamics. 2015;5:474-494.

[2] Jia Li. Discrete – time models with mosquitoes carrying genetically- modified bacteria. Mathematical Biosciences. 2012;240:35-44.

[3] RoitBerga BD, Mangelb M. Mosquito biting and movement rates as an emergent Community property and the implications for malarial interventions. Israel Journal of Ecology and Evolution. 2010;56:297–312.

[4] Jia Li. New revised simple models for interactive wild and sterile mosquito populations and their dynamics. Journal of Biological Dynamics. 2015;34:D25i92D;92D40.

[5] WHO. World malaria report summary; 2016.
Available:www.who.int/malaria/publications/world- malaria-report-2016/report/en/

[6] Bhattacharjee D, Shivaprakash G. Drug Resistance in Malaria-in a nutshell. Journal of Applied Pharmaceutical Science. 2016;6(03):137-143.

[7] Cobremeskel AA, Krogstad HE. Mathematical modeling of endemic transmission. American Journal of Applied Mathematics. 2015;3(2):36-46.

[8] Ngwa GA, Shu WS. Amathematical model for endemic with variable human and mosquito populations.
Available:www.ictp.trieste.it/pub.off.1999

[9] Olaniyi S, Olaniyi OS. Mathematical model for malaria transmission dynamics on human and mosquito population with nonlinear forces of infectious disease. Int. Journal of Pure and Applied Mathematics. 2013;88(1):125-150.

[10] Jia Li. Malaria model with stage- structured mosquitoes. Mathematical Biosciences and Engineering. 2011;8(3)753-768.

[11] Altaf Khan M, et al. Stability analysis of an SEIR model with non-linear saturated incidence and temporary immunity. Int. J. Adv. Appl. Math and Mech. 2015;2(3):234-7-2529.

[12] Shah NH, Gupta J. SEIR model and simulation for vector borne diseases. Applied Mathematics. 2013;4:13-17.

[13] Driessche PV, Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Mathematical Biosciences. 2002;180:29–48.

[14] Busenberg S, Driessche PV. Analysis of a disease transmission model in a population with varying size. J. Math. Biol. 1999;28:257-270.

[15] David R. Merkin. Introduction to the theory of stability. Springer 1997-verlag New York, Incgical.

[16] Michael YL, Wang L. A criterion for stability of matrices. Journal of Mathematical Analysis and Application. 1998;225:249-264.

[17] Smith D, et al. Ross, macdonald, and a theory for the dynamics and control of mosquito-transmitted pathogens. Ploss Pathog; 2012.
DOI: 10:371/journal.ppat.1002588

[18] Sharma S, Samanta GP. Analysis of a hand-foot-mouth disease model. International Journal of Biomathematics. 2017;10(2):1750016.

[19] Ishikawa H, et al. A mathematical model for the transmission of *Plasmodium vivax* malaria. Parasitology International. 2013;52:81-93.

[20] Macdonald G. The epidemiology and control of malaria. Oxford University Press, London; 1957.

[21] Chitnis N, Hyman JM, Cushing JM. Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. Bulletin of Mathematical Biology. 2008;70:1272-1296.

[22] Ebrahim AS-Shareef. Mathematical modelling for malaria of vertical transmission. PhD Thesis; 2015.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/22060