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Hexagonal Array Grammar System

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In 1995, J.Dassow, R.Freund, and G.Paun extended the concept of cooperating grammar system in string case to array grammars by introducing cooperating array grammar system in rectangular grids [1]. Motivated by the fact that, hexagonal arrays on triangular grid can be treated as two dimensional representation of three dimensional blocks, we extended the result of [1] to hexagonal pictures by defining hexagonal array grammar system. Context-free and regular hexagonal array grammars are two special classes of these grammars and we have made studies regarding the power of cooperation in case of hexagonal array grammars. Different types of hexagonal array grammar systems are defined and the generative capacities of these grammar systems are compared according to the number of components and modes of derivation. We observed that the difference in the generative capacity is based on the fundamental difference between regular and context free array grammars.

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1 Introduction

Cooperating string grammar systems were introduced in [2] are known to provide a formal framework for modelling distributed complex systems. The investigation of these grammar systems with respect to their generative power can be seen in [3]. In 1995, J.Dassow, R.Freund, and G.Paun extended the concept of cooperating distributed grammar system in string case to two dimensional picture description by introducing cooperating distributed array grammar system in rectangular grids [1].

Motivated by the fact that, hexagonal arrays on triangular grid can be treated as two dimensional representation of three dimensional blocks, we extend the result of [1] to hexagonal arrays. In this paper, cooperating distributed hexagonal array grammar system has been introduced and the generative power of the system has been brought out.

2 Basic Results and Definitions

For an alphabet V, by V^* we denote the monoid generated by V under the operation concatenation. That is, the set of all strings over V. The empty string is denoted by λ, and $V^* - \{\lambda\}$ is denoted by V^+. The families of regular and context free languages are denoted by REG and CF respectively. For further results of formal language theory and regulated rewriting, one can refer [4] and [5].

Now we recall the concept of cooperating distributed grammar systems and it's functioning for string case [6] and [7].

Definition 2.1. A cooperating distributed grammar system (CD grammar system in short) of degree n, $n \geq 1$ is a construct $\Gamma = (N, T, S, G_1, G_2, ..., G_n)$

where

N, T are disjoint alphabets, $S \in N$ is the axiom. $G_i = (N, T, P_i), 1 \leq i \leq n$, the so called components of the system Γ are context-free grammars (or regular grammars) without axioms where N is the non-terminal alphabet, T is the terminal alphabet, P_i is a finite set of context-free rules (or regular rules) over $N \cup T$.

Let Γ be a CD grammar system. Let $x, y \in (N \cup T)^*$, then we write $x \overset{k}{\underset{G_i}{\Longrightarrow}} y$ for $1 \leq i \leq n$, if and only if there are words $x_1, x_2, ..., x_{k+1}$
such that

(i) $x = x_1, y = x_{k+1}$

(ii) $x_j \underset{G_i}{\Longrightarrow} x_{j+1}$,

If $x_j = x'_j A_j x''_j$, then $x_{j+1} = x'_j w_j x''_j$ provided, we have the production $A_j \to w_j \in P_i$, $1 \leq j \leq k$

Moreover, we write

$$x \overset{\leq k}{\underset{G_i}{\Longrightarrow}} y \text{ if and only if } x \overset{k'}{\underset{G_i}{\Longrightarrow}} y \text{ for some } k' \leq k,$$

$x \overset{\geq k}{\underset{G_i}{\Longrightarrow}} y$ if and only if $x \overset{\leq k'}{\underset{G_i}{\Longrightarrow}} y$ for some $k' \geq k$,
$x \overset{*}{\underset{G_i}{\Longrightarrow}} y$ if and only if $x \overset{k}{\underset{G_i}{\Longrightarrow}} y$ for some k
$x \overset{t}{\underset{G_i}{\Longrightarrow}} y$ if and only if $x \overset{*}{\underset{G_i}{\Longrightarrow}} y$ and there is no $z \neq y$ with $y \overset{*}{\underset{G_i}{\Longrightarrow}} z$.

Any derivation $x \overset{k}{\underset{G_i}{\Longrightarrow}} y$ corresponds to k direct derivation steps in succession by grammars G_i, $\leq k$ derivation mode corresponds to a time limitation, since agent can perform at most k derivation steps, $\geq k$ derivation mode represent competence, since it requires agent to perform at most k derivation steps, $*$ mode denote an arbitrary derivation. And finally, t stands for a terminal derivation, where agent must perform derivation steps as long as it can.

Definition 2.2. Let Γ be a CD grammar system, and denote $D = \{*, t\} \cup$

$\{= k, \leq k, \geq k \mid k \geq 1\}$. The language generated by the system Γ in the derivation mode $F \in D$ is $L_f(\Gamma) = \{w \in t^* \mid S \Longrightarrow^f_{P_{i_1}} w_1 \Longrightarrow^f_{P_{i_2}} w_2 \Longrightarrow \cdots \Longrightarrow^f_{P_{i_m}} w_m = w$ where $m \geq 1, 1 \leq i_j \leq n, 1 \leq j \leq m\}$.

The family of languages generated by cooperating distributed grammar system with at most n components of type X in the f mode of derivation is denoted by $CD_n(X, f)$, $n \geq 1$, $f \in F$.

Array grammars are the direct extension of string grammars to two dimensional pictures consisting of symbols placed in the points with integer coordinates of the plane. Details can be found in [8]-[12].

We next recall the notion of arrays and array grammar in the sense of [9].

Definition 2.3. Let V be a finite alphabet. An array $\mathcal{A}$ over V is a function $\mathcal{A} : Z^2 \to V \cup \{\#\}$ with finite support $supp(\mathcal{A})$, where

$$supp(\mathcal{A}) = \left\{\nu \in Z^2 \mid \mathcal{A}(\nu) \neq \#\right\}$$

$\# \notin V$ is called the background or blank symbol. That is,

$$\mathcal{A} = \{(\nu, \mathcal{A}(\nu)) \mid \nu \in supp(\mathcal{A})\}.$$

The set of all arrays over V shall be denoted by V^{*2}. The empty array in V^{*2} with empty support shall be denoted by Λ. Moreover, we define $V^{+2} = V^{*2} - \{\Lambda\}$. Any subset of V^{+2} is called a (Λ-free) array language.

Definition 2.4. Let $\nu \in Z^2$. The translation $\tau_\nu : Z^2 \to Z^2$ is defined by $\tau_\nu(\omega) = \omega + \nu$ for all $\omega \in Z^2$, and for any array $\mathcal{A} \in V^{*2}$ we get $(\tau_\nu(\mathcal{A}))(\omega) = \mathcal{A}(\omega - \nu)$ for all $\omega \in Z^2$.

Definition 2.5. An array production P over V is a triple $P = (W, \mathcal{A}, \mathcal{B})$, where W is a finite subset of Z^2 and $\mathcal{A}, \mathcal{B}$ are arrays with the supports included in W. Since the pictorial representation of $\mathcal{A}$ and $\mathcal{B}$ precisely identifies the rule we can write $\mathcal{A} \to \mathcal{B}$. In the pictorial representation, all pixels of W which are not in $supp(\mathcal{A})$ will be indicated as marked by # for two arrays C,D over V and a production P as above, we write $C \Longrightarrow_P D$ if D can be obtained by replacing a subarray of C identical to $\mathcal{A}$ with $\mathcal{B}$; all pixels of W which are blank in $\mathcal{A}$ should be blank also in C.

The reflexive and transitive closure of the relation $\Longrightarrow$ is denoted by $\Longrightarrow^*$.

Definition 2.6. An array grammar is a construct,

$$G = (N, T, \#, \{(0, 0), S)\}, P)$$

where N,T are disjoint alphabets of nonterminal symbols and of terminal symbols respectively, $\# \notin N \cup T$ is the blank symbol, $S \in N$, and P is a finite set of array rewriting rules $\mathcal{A} \to \mathcal{B}$ such that at least one pixel of $\mathcal{A}$ is marked with an element of N. $\{(0,0), S)\}$ in the start array.

The array language generated by G is

$$L(G) = \{\mathcal{A} \in T^{*^2} \mid \{((0,0), S)\} \Longrightarrow^* \mathcal{A}\}.$$

Next we recall the notions of hexagonal pictures and hexagonal picture languages [13]]-[16].

Definition 2.7. A hexagonal picture P over V is a hexagonal array of symbols of V. The set of all hexagonal arrays over V is denoted by V^{**H}. A hexagonal picture language L over V is a subset of V^{**H}.

Hexagonal array grammar is a special case of array grammar, where the rules are applied in triangular grid instead of rectangular grid.

We cannot construct a hexagon using rectangles as building blocks, but we can do that using triangles. A hexagonal picture can be generated by combining six triangles. Each element in a hexagonal array using triangular grid can have six neighbours at equal distance from that element. If we are considering the arrangement of hexagon, then an element of the hexagonal picture array is the element at the middle of the hexagon and its neighbours are at each of the six corners of the hexagon.

Each symbol of a hexagonal array has exactly six neighbours, two on its own row and two each on the rows above and below it. Following figure shows a symbol 'a' in a hexagonal array and all six neighbours of 'a' represented with the symbol 'b'.

$$\begin{array}{ccccc} & b & & b & \\ b & & a & & b \\ & b & & b & \end{array}$$

Example 2.1. *A regular hexagonal array with # symbol over the alphabet $\{a\}$ is shown below*

$$\begin{array}{ccccccccccccc}
 & & & \# & & \# & & \# & & \# & & & \\
 & & \# & & a & & a & & a & & \# & & \\
 & \# & & a & & a & & a & & a & & \# & \\
\# & & a & & a & & a & & a & & a & & \# \\
 & \# & & a & & a & & a & & a & & \# & \\
 & & \# & & a & & a & & a & & \# & & \\
 & & & \# & & \# & & \# & & \# & & &
\end{array}$$

3 Isometric Hexagonal Array Grammars

An isometric array grammar was introduced by A. Rosenfeld and further studied by C.R Cook and P.S.P. Wang [17]. Using idea of isometric array in production rules, J. Dassow, R. Freund and G Paun elaborated the power of cooperation in generating two dimensional pictures [1]. In this paper, we extend the definition and results of [1] for hexagonal pictures using isometric hexagonal arrays.

We can now define an isometric hexagonal array grammar as in the case of isometric rectangular array grammars.

Definition 3.1. An isometric hexagonal array grammar is a construct

$$G = (N, T, S, P, \#)$$

where N and T are disjoint alphabets of nonterminals and terminals respectively , $S \in N$ is the start symbol, $\#$ is a special symbol called blank symbol and P is a finite set of rewriting rule of the form $\alpha \rightarrow \beta$ where α and β are finite sub patterns of a hexagonal pattern over $N \cup T \cup \{\#\}$ satisfying the following conditions

1. the shapes of α and β are identical
2. α contains at least one element of N.
3. The elements of T appearing in α are not rewritten. They remain unchanged in β.
4. The application of the production $\alpha \rightarrow \beta$ preserves connectivity of the hexagonal array.

For a hexagonal array grammar $G = (N, T, S, P, \#)$ we can define $x \Longrightarrow y$, for $x, y \in (N \cup T \cup \{\#\})^{**H}$, if there is a rule $\alpha \rightarrow \beta \in P$ such that α is a sub pattern of x and y is obtained by replacing α in x by β. The reflexive closure of $\Longrightarrow$ is denoted by $\Longrightarrow^*$. The hexagonal array language generated by G is defined by $L(G) = \{x \in T^{**H} \mid S \Longrightarrow^* x\}$.

Definition 3.2. An isometric hexagonal array grammar is said to be monotone if for all rules $\alpha \rightarrow \beta$ the non $\#$ symbol in α is not replaced by a blank symbol in β.

Definition 3.3. An isometric hexagonal array grammar is said to be context free if in the rule $\alpha \rightarrow \beta$

1. non $\#$ symbol in α are not replaced by $\#$ in β.
2. α contain exactly one nonterminal and some occurrences of blank symbol. And β contain no symbol hash.

The family of languages generated by a context free isometric hexagonal array grammar is denoted by $CFHA$.

Definition 3.4. An isometric hexagonal array grammar is said to be regular if rules are of the form

$$\begin{matrix} A & \\ & \# \end{matrix} \rightarrow \begin{matrix} a & \\ & B \end{matrix}, \quad \begin{matrix} \# & \\ & A \end{matrix} \rightarrow \begin{matrix} B & \\ & a \end{matrix}, \quad \begin{matrix} & A \\ \# & \end{matrix} \rightarrow \begin{matrix} & a \\ B & \end{matrix}, \quad \begin{matrix} & \# \\ A & \end{matrix} \rightarrow \begin{matrix} & B \\ a & \end{matrix},$$

$$A \quad \# \rightarrow a \quad B\ , \quad \# \quad A \rightarrow B \quad a\ , \ A \rightarrow a$$

The family of languages generated by a regular isometric hexagonal array grammar is denoted by $REGHA$.

All the array grammars considered in this paper are isometric, and their rules preserve the connectivity of arrays.

From the definition itself it is clear that $REGHA \subseteq CFHA$. In the next theorem we will prove that the inclusion is proper.

Theorem 3.1. $REGHA \subset CFHA$

Proof. Consider the context free hexagonal array grammar $G = (V, T, S, P, \#)$ where $N = \{S, A, B, C\}$, $T = \{a\}$,

$$P = \left\{ \begin{matrix} & \# & \\ \# & & S \\ & \# & \end{matrix} \rightarrow \begin{matrix} & B & \\ A & & a \\ & C & \end{matrix}, \quad \begin{matrix} \# & \\ & B \end{matrix} \rightarrow \begin{matrix} a & \\ & a \end{matrix}, \right.$$

$$\left. \begin{matrix} & C \\ \# & \end{matrix} \rightarrow \begin{matrix} & a \\ a & \end{matrix}, \quad \# \quad A \rightarrow A \quad a\ , A \rightarrow a \right\}$$

It will generate the language L, which is the set of all right arrows over one letter alphabet of the form

$$\begin{array}{cccccc} & & & a & & \\ & & & & a & \\ a & a & \cdots & a & & a \\ & & & & a & \\ & & & a & & \end{array}$$

Therefore $L \in CFHA$

Suppose L can be generated by a regular hexagonal array grammar. In a regular hexagonal array grammar in any stage of derivation the sentential form will contain only one nonterminal. In other words it can grow only in one direction. Now let us assume that the derivation of an element in L starts from the tail of the arrow. Then at some point it should grow towards the left or right arrow head. But then it cannot come back to the other side of the arrow head. Therefore it is not possible to generate L using a regular hexagonal array grammar. That is, $L \notin REGHA$

Therefore the inclusion is proper. $REGHA \subset CFHA$. □

4 Cooperating Distributed Hexagonal Array Grammar System

The definition of cooperating distributed hexagonal array grammar system is obtained by simply considering sets of context free (respectively regular) hexagonal array rewriting rules instead of context free (respectively regular) string rewriting rules in cooperating grammar systems. Now we give formal definition of cooperating distributed hexagonal array grammar system.

Definition 4.1. A cooperating distributed hexagonal array grammar system (of type X, $X \in \{CFHA, REGHA\}$, and of degree n,$n \geq 1$), is a construct

$$\Gamma = (N, T, S, P_1, P_2, \ldots, P_n, \#)$$

where N, T are non-terminal alphabet and terminal alphabet respectively, $S \in N$ is the starting symbol, $P_1, P_2, \ldots, P_n$ are finite sets of regular respectively context free rules over $N \cup T$ and # is the blank symbol used in isometric hexagonal array grammars.

Definition 4.2. Let Γ be a cooperating distributed hexagonal array grammar system. Let $x, y \in T^*$. Then we write $x \Longrightarrow_{P_i}^{k} y$ if and only if there are words $x_1, x_2, \cdots, x_{k+1}$ such that

1. $x = x_1$, $y = x_{k+1}$,
2. $x_j \Rightarrow_{P_i} x_{j+1}$, That is, $x_j = x_j{}' A_j x_j{}''$, $x_{j+1} = x_j{}' w_j x_j{}''$, $A_j \rightarrow w_j \in P_i$, $1 \leq j \leq k$.

Moreover, we write

$x \Rightarrow_{P_i}^{\leq k} y$ if and only if $x \Rightarrow_{P_i}^{k'} y$ for some $k' \leq k$,
$x \Rightarrow_{P_i}^{\geq k} y$ if and only if $x \Rightarrow_{P_i}^{k'} y$ for some $k' \geq k$,
$x \Rightarrow_{P_i}^{*} y$ if and only if $x \Rightarrow_{P_i}^{k} y$ for some k,
$x \Rightarrow_{P_i}^{t} y$ if and only if $x \Rightarrow_{P_i}^{*} y$ and there is no $Z \neq y$ with $y \Rightarrow_{P_i}^{*} z$.

By $CD_n(X, f)$ we denote family of hexagonal array language generated by cooperating distributed hexagonal array grammar system consisting of at most n components of type $X \in \{REGHA, CFHA\}$ in the mode f.

To show the power of a cooperating distributed hexagonal array grammar system, consider the following example of a language of set of all regular hexagons over a one letter alphabet.

Example 4.1. $\Gamma = (N, \{a\}, S, P_1, P_2, P_3, \#)$

where $N = \{S, A, B, C, D, E, F, A', B', C', D', E', F', A'', B'', C'', D'', E'', F''\}$

$$P_1 = \left\{ \begin{array}{ccccc} & \# & & \# & \\ \# & & S & & \# \\ & \# & & \# & \end{array} \rightarrow \begin{array}{ccccc} & a & & a & \\ a & & a & & a \\ & a & & a & \end{array},\ \begin{array}{ccccc} & \# & & \# & \\ \# & & S & & \# \\ & \# & & \# & \end{array} \rightarrow \begin{array}{ccccc} & A & & B & \\ F & & a & & C \\ & E & & D & \end{array}, \right.$$

$$\begin{array}{ccc} & \# & \\ \# & & A \end{array} \rightarrow \begin{array}{ccc} & G' & \\ A' & & a \end{array},\ \begin{array}{ccc} \# & & \# \\ & B & \end{array} \rightarrow \begin{array}{ccc} B' & & H' \\ & a & \end{array},$$

$$\begin{array}{ccc} & \# & \\ C & & \# \end{array} \rightarrow \begin{array}{ccc} & C' & \\ a & & I' \end{array},\ \begin{array}{ccc} D & & \# \\ & \# & \end{array} \rightarrow \begin{array}{ccc} a & & D' \\ & J' & \end{array},$$

$$\begin{array}{ccc} & E & \\ \# & & \# \end{array} \rightarrow \begin{array}{ccc} & a & \\ K' & & E' \end{array},\ \begin{array}{ccc} \# & & F \\ & \# & \end{array} \rightarrow \begin{array}{ccc} L' & & a \\ & F' & \end{array},$$

$$\begin{array}{cc} \# & \\ & G \end{array} \rightarrow \begin{array}{cc} G' & \\ & a \end{array},\ \begin{array}{cc} & \# \\ H & \end{array} \rightarrow \begin{array}{cc} & H' \\ a & \end{array},\ I\ \# \rightarrow a\ I',$$

$$\begin{array}{cc} J & \\ & \# \end{array} \rightarrow \begin{array}{cc} a & \\ & J' \end{array},\ \begin{array}{cc} & K \\ \# & \end{array} \rightarrow \begin{array}{cc} & a \\ K' & \end{array},\ \#\ L \rightarrow L'\ a,$$

$$\left. \begin{array}{l} A'' \rightarrow A, B'' \rightarrow B, C'' \rightarrow C, D'' \rightarrow D, E'' \rightarrow E, F'' \rightarrow F, \\ G'' \rightarrow G, H'' \rightarrow H, I'' \rightarrow I, J'' \rightarrow J, K'' \rightarrow K, L'' \rightarrow L \end{array} \right\}$$

$$P_2 = \left\{ \begin{array}{l} A' \rightarrow A'', B' \rightarrow B'', C' \rightarrow C'', D' \rightarrow D'', E' \rightarrow E'', F' \rightarrow F'' \\ G' \rightarrow G'', H' \rightarrow H'', I' \rightarrow I'', J' \rightarrow J'', K' \rightarrow K'', L' \rightarrow L'' \end{array} \right\}$$

$$P_3 = \left\{ \begin{array}{l} A'' \rightarrow a, B'' \rightarrow a, C'' \rightarrow a, D'' \rightarrow a, E'' \rightarrow a, F'' \rightarrow a \\ G'' \rightarrow a, H'' \rightarrow a, I'' \rightarrow a, J'' \rightarrow a, K'' \rightarrow a, L'' \rightarrow a \end{array} \right\}$$

For illustrating the work of the system consider the derivation of a regular hexagon with sides of length three unit:

$$\begin{array}{ccccc} & \# & & \# & \\ \# & & S & & \# \\ & \# & & \# & \end{array} \Longrightarrow_t^{P_1} \begin{array}{ccccc} & A & & B & \\ F & & a & & C \\ & E & & D & \end{array}$$

$$\Longrightarrow_t^{P_1} \begin{array}{ccccccccc} & & G' & & B' & & H' & & \\ & A' & & a & & a & & C' & \\ L' & & a & & a & & a & & I' \\ & F' & & a & & a & & D' & \\ & & K' & & E' & & J' & & \end{array}$$

$$\Longrightarrow_t^{P_2} \begin{array}{ccccccccc} & & G'' & & B'' & & H'' & & \\ & A'' & & a & & a & & C'' & \\ L'' & & a & & a & & a & & I'' \\ & F'' & & a & & a & & D'' & \\ & & K'' & & E'' & & J'' & & \end{array}$$

$$\Longrightarrow_t^{P_3} \begin{array}{ccccccccc} & & a & & a & & a & & \\ & a & & a & & a & & a & \\ a & & a & & a & & a & & a \\ & a & & a & & a & & a & \\ & & a & & a & & a & & \end{array}$$

Therefore Γ *generates all regular hexagons over* $\{a\}$.

Lemma 4.2. *Let LA be the set of all left arrow heads over one letter alphabet of the form*

$$\begin{array}{ccccc} & & a & & a \\ & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \\ a & & a & & \\ & \ddots & & \ddots & \\ & & a & & a \end{array}$$

Then $LA \in CD_3(CFA, t)$.

Proof. We consider the system

$$\Gamma = (\{S, A, B, C, D, A^{'}, B^{'}, C', D'\}, \{a\}, S, P_1, P_2, P_3, \#)$$

with set of productions,

$$P_1 = \left\{ \begin{array}{ccc} & \# & \\ S & & \# \\ & \# & \end{array} \rightarrow \begin{array}{ccc} & A^{'} & \\ a & & a \\ & B^{'} & \end{array} , \begin{array}{cc} & \# \\ A & \end{array} \rightarrow \begin{array}{cc} & A^{'} \\ a & \end{array} , \begin{array}{cc} B & \\ & \# \end{array} \rightarrow \begin{array}{cc} a & \\ & B^{'} \end{array} , C \rightarrow a, D \rightarrow a \right\}$$

$$P_2 = \left\{ A^{'} \rightarrow A \, , B^{'} \rightarrow B \, , \begin{array}{cc} & C \\ \# & \end{array} \rightarrow \begin{array}{cc} & a \\ C & \end{array} , \begin{array}{cc} \# & \\ & D \end{array} \rightarrow \begin{array}{cc} D & \\ & a \end{array} , C' \rightarrow C, D' \rightarrow D \right\}$$

$$P_3 = \{ \; A^{'} \;\; \# \; \rightarrow \; a \;\; C' \; , \; B^{'} \;\; \# \; \rightarrow \; a \;\; D' \; \}$$

Consider the derivation of an element of LA for illustrating the work of the above system:

$$\begin{array}{ccc} & \# & \\ S & & \# \\ & \# & \end{array} \Longrightarrow_{P_1} \begin{array}{ccc} & A^{'} & \\ a & & a \\ & B^{'} & \end{array} \Longrightarrow_{P_2} \begin{array}{ccc} & A & \\ a & & a \\ & B & \end{array} \Longrightarrow_{P_1} \begin{array}{ccc} & & A^{'} \\ & a & \\ a & & a \\ & a & \\ & & B^{'} \end{array} \Longrightarrow_{P_3}$$

$$\begin{array}{cccc} & & a & C' \\ & a & & \\ a & & a & \\ & a & & \\ & & a & D' \end{array} \Longrightarrow_{P_2} \begin{array}{ccccc} & & a & & a \\ & a & & C & \\ a & & a & & \\ & a & & D & \\ & & a & & a \end{array} \Longrightarrow_{P_1} \begin{array}{ccccc} & & a & & a \\ & a & & a & \\ a & & a & & \\ & a & & a & \\ & & a & & a \end{array}$$

In general, in the $t - mode$ derivation in Γ runs as follows : Starting from S, the nonterminals $A^{'}$, $B^{'}$ introduced are growing in north east direction and south west direction respectively using the rules from P_1 and P_2. At some point of time it uses the rules from P_3 and put an a in the horizontal direction in both ends. Then it uses the rules from P_2 and grow in the south west and

north west direction until it reaches a. When both the terminals reaches a the system will switch over to the component P_1 and replaces the nonterminals by a.
Therefore $LA \in CD_3(CFA, t)$ □

Lemma 4.3. *$LA \notin CD_n(REGHA, t)$.*

Proof. Assume that $LA = L_t(\Gamma)$ for some regular system

$$\Gamma = (N, \{a\}, S, P_1, P_2, \cdots, P_n, \#)$$

As the rules in the components are regular the derivation must start in some point marked by S and proceeds along the hexagonal contour until completing it. An element of LA contains two pairs of parallel edges except the horizontal ones. Consider an arrow head with one slanting edge has length m. Assume m is larger than the cardinality of N. During the derivation of that edge of length m we can find a derivation of the form $\begin{matrix} & \# \\ A & \end{matrix} \Rightarrow^* \begin{matrix} & A \\ u & \end{matrix}$ such that $|u| \geq 1$ and the nonterminal A is rewritten in the same component of Γ. And during the generation of its parallel edge we can find a derivation of the form $\begin{matrix} & B \\ \# & \end{matrix} \Rightarrow^* \begin{matrix} & v \\ B & \end{matrix}$ such that $|v| \geq 1$ and the nonterminal B is rewritten in the same component of Γ. Now by iterating the first derivation $|v|$ times and second derivation $|u|$ times we get an arrow head with upper part has length $m + |u|.|v|$ and lower part of the arrow head has length m. Therefore $LA \notin CD_n(REGHA, t)$. □

Lemma 4.4. *For $n \geq 1, X \in \{REGHA, CFHA\}$ and $f \in \{*, t\} \cup \{\leq k, = k, \geq k \mid k \geq 1\}, f' \in \{*, t, = 1, \geq 1\} \cup \{\leq k \mid k \geq 1\}$*

1. *$CD_n(X, f) \subseteq CD_{n+1}(X, f)$.*
2. *$CD_n(REGHA, f) \subseteq CD_n(CFHA, f)$.*
3. *$CFHA \subseteq CD_1(CFHA, f)$ and $REGHA \subseteq CD_1(REGHA, f')$.*

Proof. (1) and (2) are obvious from the definitions.

In the case of (3) for every mode in f' consider the grammar
$G = (N, T, P, S, \#)$ as a cooperating system with one component ($G \in REGHA$ or $G \in CFHA$). For $\geq k$ mode and $= k$ mode, in case of $CFHA$ consider the cooperating grammar system $\Gamma = (N, T, S, P \cup \{A \to A | A \in N\}, \#)$. The rules $A \to A$ ensure the fact that Γ can perform derivations of arbitrary length, hence we have $L(G) = L(\Gamma)$. Therefore $CFHA \subseteq CD_1(CFHA, f)$. □

Lemma 4.5. *For $n \geq 1$, $X \in \{REGHA, CFHA\}$, $f \in \{*, = 1, \geq 1\} \cup \{\leq k | k \geq 1\}$, $CD_n(X, f) \subseteq X$.*

Proof. For the cooperating distributed hexagonal array grammar system
$\Gamma = (N, T, S, P_1, P_2, \cdots, P_n, \#)$ construct the hexagonal array grammar
$G = (N, T, S, \bigcup_{i=1}^{n} P_i, \#)$ and we have $L_f(\Gamma) = L(G)$. □

Theorem 4.6. *For $n \geq 1$, $X \in \{REGHA, CFHA\}$, $f \in \{*, = 1, \geq 1\} \cup \{\leq k | k \geq 1\}$, $CD_n(X, f) = X$.*

Proof. The result follows from Lemma 4.4 and Lemma 4.5. Hence it established the solution for the derivation modes $= 1$ and ≥ 1. □

Now we can consider the t-mode of derivation

Lemma 4.7. $CD_1(X,t) \subseteq X, X \in \{REGHA, CFHA\}$.

Proof. The cooperating distributed hexagonal array grammar system with one component working in $t - mode$ is same as a hexagonal array grammar. □

Lemma 4.8. $CD_2(REGHA,t) - CFHA \neq \Phi$

Proof. We will prove the theorem by using the following example. The set of all hollow parallelogram can be generated by the following regular hexagonal array grammar system with two components.

$$G = (\{S, A, A', B, B', C, C', D, Z\}, \{a\}, S, P_1, P_2, \#)$$

$$P_1 = \left\{ \begin{array}{cc} & \# \\ S & \end{array} \rightarrow \begin{array}{cc} & S \\ a & \end{array},\ \begin{array}{cc} & \# \\ S & \end{array} \rightarrow \begin{array}{cc} & A' \\ a & \end{array},\ A' \ \# \rightarrow a \ A, \right.$$
$$A \ \# \rightarrow a \ A,\ A \ \# \rightarrow a \ B',\ \begin{array}{cc} & B' \\ \# & \end{array} \rightarrow \begin{array}{cc} & a \\ B & \end{array},$$
$$\begin{array}{cc} & B \\ \# & \end{array} \rightarrow \begin{array}{cc} & a \\ B & \end{array},\ \begin{array}{cc} & B \\ \# & \end{array} \rightarrow \begin{array}{cc} & a \\ C' & \end{array},\ \# \ C' \rightarrow C \ a,$$
$$\left. \# \ C \rightarrow C \ a,\ \# \ C \rightarrow D \ a,\ \begin{array}{cc} & \# \\ D & \end{array} \rightarrow \begin{array}{cc} & Z \\ a & \end{array} \right\}$$

$P_2 = \{D \rightarrow a\}$

For illustrating the work of the system consider the derivation of a parallelogram:

$$\begin{array}{cc} & \# \\ S & \end{array} \Longrightarrow^{P_1} \begin{array}{cc} & S \\ a & \end{array} \Longrightarrow^{P_1} \begin{array}{ccc} & & A' \\ & a & \\ a & & \end{array} \Longrightarrow^{P_1} \begin{array}{cccc} & & a & A \\ & a & & \\ a & & & \end{array}$$

$$\Longrightarrow^{P_1} \begin{array}{ccccc} & & a & a & B' \\ & a & & & \\ a & & & & \end{array} \Longrightarrow^{P_1} \ldots \Longrightarrow^{P_1} \begin{array}{ccccccc} & & a & & a & & a \\ & a & & & & a & \\ a & & & & B & & \end{array}$$

$$\Longrightarrow^{P_1} \begin{array}{ccccccc} & & a & & a & & a \\ & a & & & & a & \\ a & & & & a & & \\ & & & C' & & & \end{array} \Longrightarrow^{P_1} \begin{array}{ccccccc} & & a & & a & & a \\ & a & & & & a & \\ a & & & & a & & \\ & C & & a & & & \end{array}$$

$$\ldots \Longrightarrow^{P_1} \begin{array}{cccccccc} & & & a & & a & & a \\ & & a & & & & a & \\ & a & & & & a & & \\ D & & a & & a & & & \end{array} \Longrightarrow^{P_2} \begin{array}{cccccccc} & & & a & & a & & a \\ & & a & & & & a & \\ & a & & & & a & & \\ a & & a & & a & & & \end{array}$$

For proving the above language does not belongs to $CFHA$, let us start by assuming that it can be generated by a context free hexagonal array grammar. Since a context free hexagonal array grammar can have more than one 'growing heads' let us assume that from the start symbol it starts generating two sides of the parallelogram. And the nonterminals on each hand should occur on the end of the lines. To complete the derivation of the parallelogram the two growing heads should meet at some point. And then it should change to the terminal. But the sequence of rules to change the nonterminal to the terminal can apply before the meeting of the two heads also since we can use the rules arbitrarily. But then it cannot complete the parallelogram. Therefore the above language does not belongs to $CFHA$. □

Lemma 4.9. $CFHA - CD_n(REGHA, f) \neq \Phi$.

Proof. To prove the theorem consider the context free hexagonal array language we consider in lemma 4.2. But that language cannot be generated by regular hexagonal grammar system. Because the rules of a regular hexagonal array grammar contains only one growing head. But for generating the arrow head of the patterns of the language we should require two growing heads at the same time. □

From the above two lemmas it is clear that $CD_n(REGHA, f)$ and $CFHA$ are not comparable, where f can be any mode.

5 Conclusion

In this paper, we have introduced the cooperating distributed hexagonal array grammar system and studied about the generative power of it. We observe that systems with only two regular hexagonal array components and with t mode of derivation can generate sets of arrays which cannot be described by context free hexagonal array grammars, which is a situation contradicting the corresponding results in string grammar systems.

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Competing Interest

The authors declare that no competing interests exist.

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