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A Few Results on Wiener Index of the kth Power of Some Specific Graphs

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Abstract

For a simple connected undirected graph $G = (V, E)$, the $Wiener\ index$ $W(G)$ of G is defined as half the sum of the shortest-path distances between all pairs of vertices u, v of G. The kth power of a graph G, denoted by G^k, is a graph with the same vertex set as G such that two vertices are adjacent in G^k if and only if their distance is at most k in G. Let P_n be a path on n vertices. In this paper, for the graph G = $P_n 2P_n$, we obtain a closed form expression for $W(G^2)$. In addition, a correct closed form expression is stated for $W(P_n^3)$. But we are unable to provide a proof for $W(P_n^3)$ of how such expression has arrived. This may be compared with the existing result: for a graph G = $P_n 2P_n$, $W(G^2)$ can be computed by an algorithm in linear time.

Keywords: Average distance; distance in graphs; graph algorithms; kth power of a graph; Wiener index.

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1 Introduction

Let $G = (V(G), E(G))$ be a finite connected unweighted undirected graph without self-loops and multiple edges. Let $|V(G)| = n$ and $|E(G)| = m$ denote the $order$ and $size$ of a graph G respectively. A sequence $(u = v_1, v_2, \ldots, v_l = v)$ of pairwise distinct vertices is a $u - v\ path$ in G if $u = v_1v_2, \ldots, v_{l-1}v_l = v \in E(G)$. The length of the $u - v$ path is the number of edges on that path. The path of order n is denoted by P_n. The $distance\ d_G(u, v)$ (or simply $d(u, v)$) between two vertices u and v is the length of a shortest $u - v$ path. The $all - pairs\ shortest - lengths$ (APSL) problem is to find $d(u, v)$ for all pairs of vertices $u, v \in V(G)$. The $Wiener\ index\ W(G)$ of G is defined as

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v). \tag{1.1}$$

Wiener index is a distance based graph invariant which is one of the most popular topological indices in mathematical chemistry. It is named after the chemist Harold Wiener, who first introduced it in 1947 to study chemical properties of alkanes. It is now recognized that there are good correlations between $W(G)$ and physico-chemical properties of the organic compound from which G is derived. For applications of Wiener index, see [1,2]. A related quantity is the $average\ distance\ \mu(G)$ defined as

$$\mu(G) = \frac{2W(G)}{n(n-1)}\ .$$

When G represents a network, $\mu(G)$ can be viewed as a measure of the average delay of messages to traverse between two nodes of the network.

As Nishimura et al. mentions in [3], one of the basic problems in computational biology is the reconstruction of the phylogeny, or evolutionary history, of a set of species or genes, typically represented as a phylogenetic tree, whose leaves are a distinct known species. The authors in [3] views correlations between the problem of forming phylogenetic tree and the problem of forming a tree from a graph. One such correlation between graphs and trees, or more generally between graphs and graphs, arises in the notion of graph powers. The kth power of a graph, $G^k = (V(G), E(G^k))$ is a graph with the same vertex set as G such that two vertices are adjacent in G^k if and only if their distance is at most k in G. Thus in such applications computation of distance plays a crucial role.

The $Cartesian\ product$ of two graphs G and H denoted by $G2H$ is the graph with vertex set $V(G) \times V(H)$. Every vertex of $G2H$ is thus an ordered pair (u, v), where $u \in V(G)$ and $v \in V(H)$. Two distinct vertices (u, v) and $(u^{'}, v^{'})$ are adjacent in $G2H$ if either (1) $u = u^{'}$ and $vv^{'} \in E(H)$, or (2) $v = v^{'}$ and $uu^{'} \in E(G)$.

As an operation of graphs, the Cartesian product is an important method in constructing very large scale networks from several small graphs. As Xu $at\ al.$ mentions in [4], the graph that is constructed from several small graphs can contain the factor graphs (a $factor\ graph$ is a bipartite graph that expresses how a global function of many variables factors into a product of local functions) as subgraphs which preserves some properties of the factor graphs, such as regularity, vertex-transitivity, hamiltonicity, and so forth. Interestingly, from factor graphs we can easily compute some important parameters of a graph such as diameter, degree and connectivity. Thus, the Cartesian product plays an important role in the design and analysis of large scale computer systems and interconnection networks [5]. For example, hypercube of dimension n, Q_n, is the Cartesian product of n copies of K_2, where K_2 is an edge.

Entringer $et\ al.$ [6], obtained the closed form expressions for computing Wiener index of large classes of trees. Among all trees of order n, the best known are $W(P_n) = \binom{n+1}{3}$ and $W(S_n) = (n-1)^2$, where P_n and S_n denote the $path$ and $star$ of a graph of order n respectively. The authors showed that:

Theorem 1.1. *Let T_n be any tree of order n that is different from P_n and S_n. Then $W(S_n) \leq W(T_n) \leq W(P_n)$.*

In a manner similar to the bound given in Theorem 1.1, it also holds for the kth power of a graph. The following theorem and corollary can be found in [7].

Theorem 1.2. *Let T_n^k be the kth power of any tree of order n. Then $W(S_n^k) \leq W(T_n^k) \leq W(P_n^k)$.*

Corollary 1.3. *For a connected undirected graph G of order n, $W(G^k) \leq W(P_n^k)$.*

Motivated by the background of Wiener index, average distance, notion of graph powers, Cartesian product operation, and the results above, in this paper we obtain a closed-form expression for $W(G^2)$, where G = $P_n 2 P_n$. Subsequently, we also obtain a correct closed form expression for $W(P_n^3)$. But we are unable to provide a proper theoretical proof of how an expression has arrived for $W(P_n^3)$. This may be compared with the existing result: for a graph G = $P_n 2 P_n$, $W(G^2)$ can be computed by an algorithm in linear time [8].

2 Main Results

We begin with the following lemma.

Lemma 2.1. *Let P_n be a path on n vertices. Then*

$$W(P_n^k 2 P_n^k) \;=\; 2n^2 W(P_n^k). \tag{2.1}$$

Proof. The lemma is a consequence of the following result proved in [9]. Let G and H be two given graphs. Then

$$W(G2H) \;= \mid V(H) \mid^2 W(G) + \; \mid V(G) \mid^2 W(H). \quad \square$$

Theorem 2.2. *For a graph $P_n^k 2 P_n^k$ of order n^2, $W(P_n^k 2 P_n^k)$ can be computed algorithmically in time $O(n)$.*

Proof. From (2.1), the idea is to first compute $W(P_n^k)$ and then $W(P_n^k 2 P_n^k)$ = $2n^2 W(P_n^k)$. From [7], it has been shown that $W(P_n^k)$ = $\sum_{j=1}^{n-1} \lceil \frac{j}{k} \rceil (n-j)$. Hence it follows that $W(P_n^k)$ can be computed in $O(n)$. $\square$

Theorem 2.3. *Let $G = P_n 2 P_n$, where P_n is a path on n vertices. Then*

$$W(G^2) = \begin{cases} \frac{1}{24}\left(4n^5 \;+\; 3n^4 \;-\; 4n^3\right) & \text{if } n \text{ is even,} \\ \frac{1}{24}\left(4n^5 \;+\; 3n^4 \;-\; 4n^3 \;-\; 3\right) & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

Proof. Let H = $P_n^2 2 P_n^2$ be a graph of order n^2. Let the vertices of graphs G^2 and H be arranged in two dimensions of n-rows and n-columns. Also, let the vertices be numbered 1, 2, $\ldots$, n^2, where the vertex number of ith-row and jth-column in a grid represent $(i-1)n + j$ for $1 \leq i, j \leq n$. For $u \in V(G)$ and $Q \subset V(G)$, let $d^+(u, Q)$ = $\sum_{v \in Q} d(u, v)$. In G^2, a cross edge is defined as follows: an edge $(u, v) \in E(G^2)$ is a $cross$ edge if $d(u, v)$ = 2, and uth-row $\neq$ vth-row, uth-column $\neq$ vth-column. It may be noted that since G^2 is undirected, in each row of vertices (first-row to $(n-1)$th-row) we consider a cross edge (u, v) such that it is an edge connecting a vertex u to a vertex v, $u < v$. We know that $V(H)$ = $V(G^2)$ and $E(H) \subset E(G^2)$. So $E(G^2)$ = $E(H) \cup CE$, where CE = $E(G^2) \backslash E(H)$, the set of cross edges in G^2. Since $V(H)$ = $V(G^2)$ and $E(H) \subset E(G^2)$, $W(G^2)$ can be computed using the result of $W(H)$ as shown below:
Let X_{1j} = $V(G^2) \backslash \{1, \ldots j\}$. We consider two cases A and B.

$Case\ A$**:** n is even. Consider the first-row vertices 1, 2, $\ldots$, n of G^2. Let RE_1 = $\{$(1, $n+2$), (2, $n+1$), (2, $n+3$), (3, $n+2$), (3, $n+4$), $\ldots$, $(n-1, 2n-2)$, $(n-1, 2n)$, $(n, 2n-1)\}$ be the set of cross edges for first-row vertices of G^2. From the set RE_1, we can see that in G^2 there is a single cross edge that is incident on vertices 1 and n, and there are two cross edges incident on each of the vertices 2, 3, $\ldots$, $n-1$. We now compute $d^+_{G^2}(1, X_{11})$, $d^+_{G^2}(i, X_{1i})$ $(2 \leq i \leq n-1)$, and $d^+_{G^2}(n, X_{1n})$ as follows:

1. Computation of $d^+_{G^2}(1, X_{11})$: Let x be the number of edges incident on vertex 1 in H. Then there are $x + 1$ edges incident on vertex 1 in G^2. That is, the extra edge that is incident on vertex 1 in G^2 is $(1, n + 2)$. Thus $d^+_{G^2}(1, X_{11}) < d^+_H(1, X_{11})$ and $d^+_{G^2}(1, X_{11})$ can be obtained by reducing the distance $\left(\frac{n}{2}\right)^2$ from $d^+_H(1, X_{11})$. This is possible because $(1, n + 2) \in E(G^2)$ and hence we can reduce a unit distance to $\frac{n}{2}$ number of vertices in one row of a grid and there are $\frac{n}{2}$ such rows in a grid. That is, we can reduce a unit distance from vertex 1 to each of the vertices given in the following sets.
$\{n + 2, n + 4, n + 6, \ldots, 2n - 2, 2n\} \cup \{3n + 2, 3n + 4, 3n + 6, \ldots, 4n - 2, 4n\} \cup \ldots \cup \{(n - 1)n + 2, (n - 1)n + 4, (n - 1)n + 6, \ldots, n^2 - 2, n^2\}$. It is clear that there are $\frac{n}{2}$ vertices in each set and there are $\frac{n}{2}$ such sets. Thus

$$d^+_{G^2}(1, X_{11}) \;=\; d^+_H(1, X_{11}) \;-\; \left(\frac{n}{2}\right)^2. \tag{2.3}$$

In a manner similar to $d^+_{G^2}(1, X_{11})$, we compute $d^+_{G^2}(n, X_{1n})$ as

$$d^+_{G^2}(n, X_{1n}) \;=\; d^+_H(n, X_{1n}) \;-\; \left(\frac{n}{2}\right)^2. \tag{2.4}$$

2. Computation of $d^+_{G^2}(i, X_{1i})$ $(2 \le i \le n - 1)$: The number of edges incident on vertex i in G^2 is $x + 2$, where x is the number of edges incident on vertex i in H. The extra two edges that is incident on vertex i in G^2 are the cross edges $(i, n + (i - 1))$ and $(i, n + (i + 1))$. Thus similar to item 1, with edges $(i, n + (i - 1))$ and $(i, n + (i + 1))$ in G^2 and not in H, $d^+_{G^2}(i, X_{1i})$ can be obtained by reducing the distance $\left(\frac{n}{2}\right)^2$ from $d^+_H(i, X_{1i})$. Thus
$$d^+_{G^2}(i, X_{1i}) \;=\; d^+_H(i, X_{1i}) \;-\; \left(\frac{n}{2}\right)^2. \tag{2.5}$$

Now we add (2.3), (2.4) and (2.5), and let this result be D^e_1.

$$\begin{aligned} D^e_1 &= d^+_{G^2}(1, X_{11}) + d^+_{G^2}(2, X_{12}) + \ldots + d^+_{G^2}(n-1, X_{1n-1}) + d^+_{G^2}(n, X_{1n}) \\ &= d^+_H(1, X_{11}) - \left(\frac{n}{2}\right)^2 + d^+_H(2, X_{12}) - \left(\frac{n}{2}\right)^2 + \;\ldots\; + d^+_H(n, X_{1n}) - \left(\frac{n}{2}\right)^2 \\ &= \left[d^+_H(1, X_{11}) \;+\; d^+_H(2, X_{12}) \;+\; \ldots \;+\; d^+_H(n, X_{1n})\right] \;-\; n\left(\frac{n}{2}\right)^2. \end{aligned} \tag{2.6}$$

Let $RE_2 = \{(n + 1, 2n + 2), (n + 2, 2n + 1), (n + 2, 2n + 3), \ldots, (2n - 1, 3n - 2), (2n - 1, 3n), (2n, 3n - 1)\}$ be the set of cross edges for second-row vertices of G^2. In a manner similar to (2.6) with RE_2 in G^2, we add $d^+_{G^2}(n + 1, X_{1n+1})$, $d^+_{G^2}(n + 2, X_{1n+2})$, $\ldots$, $d^+_{G^2}(2n, X_{12n})$, and let this result be D^e_2.

$$\begin{aligned} D^e_2 \;&= d^+_{G^2}(n+1, X_{1n+1}) + d^+_{G^2}(n+2, X_{1n+2}) \;+ \ldots \;+ d^+_{G^2}(2n, X_{12n}) \\ &= \; d^+_H(n+1, X_{1n+1}) \;-\; \left(\frac{n}{2} - 1\right)\frac{n}{2} \;+\; d^+_H(n+2, X_{1n+2}) \;-\; \left(\frac{n}{2} - 1\right)\frac{n}{2} \\ &\quad + \;\ldots\; + \; d^+_H(2n, X_{12n}) \;-\; \left(\frac{n}{2} - 1\right)\frac{n}{2} \\ &= \; \left[d^+_H(n+1, X_{1n+1}) \;+\; \ldots \;+\; d^+_H(2n, X_{12n})\right] \;-\; n\left[\left(\frac{n}{2} - 1\right)\frac{n}{2}\right]. \end{aligned} \tag{2.7}$$

Similar to (2.6) and (2.7), for an ith-row vertices of G^2, where $i = 4, 6, 8, \ldots, n - 4, n - 2$, we can determine D^e_4, D^e_6, $\ldots$, D^e_{n-2}. Notice that $D^e_3 = D^e_2$, $D^e_5 = D^e_4$, $\ldots$, $D^e_{n-1} = D^e_{n-2}$. Let $D^e_n = d^+_{G^2}((n-1)n+1, X_{1(n-1)n+1}) + d^+_{G^2}((n-1)n+2, X_{1(n-1)n+2}) + \ldots + d^+_{G^2}(n^2-2, X_{1n^2-2}) + d_{G^2}(n^2 - 1, n^2)$. Note that since there are no cross edges for nth-row vertices in G^2, the distance D_n^e is equal to both the graphs G^2 and H. Thus adding $D_1^e, D_2^e, \ldots, D_{n-1}^e$ and D_n^e, we get $W(G^2)$ as

$$\begin{aligned} W(G^2) &= \left[d_H^+(1, X_{11}) + d_H^+(2, X_{12}) + \ \ldots \ + d_H^+((n-1)n, X_{1(n-1)n}) + D_n^e\right] - \\ &\left[n\left(\frac{n}{2}\right)^2 + 2\left(n\left(\frac{n}{2}-1\right)\frac{n}{2}\right) + 2\left(n\left(\frac{n}{2}-2\right)\frac{n}{2}\right) + \ldots + 2\left(n\left(\frac{n}{2}\right)\right)\right] \\ &= W(H) - \frac{n^4}{8}, \end{aligned} \tag{2.8}$$

where

$$W(H) = \left[d_H^+(1, X_{11}) + d_H^+(2, X_{12}) + \ \ldots \ + d_H^+((n-1)n, X_{1(n-1)n}) + D_n^e\right].$$

From (2.1) for k = 2, we have $W(H) = 2n^2 W(P_n^2)$. In [7] it has been shown that

$$W(P_n^2) = \frac{1}{2}\left(\frac{n^3-n}{6} + \lfloor\frac{n^2}{4}\rfloor\right). \tag{2.9}$$

Thus

$$W(H) = n^2\left(\frac{n^3-n}{6} + \lfloor\frac{n^2}{4}\rfloor\right). \tag{2.10}$$

Substituting (2.10) in (2.8), we get

$$W(G^2) = n^2\left(\frac{n^3-n}{6}\right) + \frac{n^4}{8}. \tag{2.11}$$

Simplifying (2.11), we get

$$W(G^2) = \frac{1}{24}\left(4n^5 + 3n^4 - 4n^3\right). \tag{2.12}$$

$Case\ B$**:** n is odd. First, we compute $d_{G^2}^+(i, X_{1i})$ $(1 \leq i \leq n)$ as follows:
In a manner similar to (2.3), when i is odd, we get

$$d_{G^2}^+(i, X_{1i}) = d_H^+(i, X_{1i}) - \left(\lfloor\frac{n}{2}\rfloor\right)^2,$$

and when i is even, we get

$$d_{G^2}^+(i, X_{1i}) = d_H^+(i, X_{1i}) - \left(\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right). \tag{2.13}$$

Now we add $d_{G^2}^+(1, X_{11}), d_{G^2}^+(2, X_{12}), \ldots, d_{G^2}^+(n, X_{1n})$, and let this result be D_1^o.

$$\begin{aligned} D_1^o &= d_{G^2}^+(1, X_{11}) + d_{G^2}^+(2, X_{12}) + \ldots + d_{G^2}^+(n-1, X_{1n-1}) + d_{G^2}^+(n, X_{1n}) \\ &= d_H^+(1, X_{11}) - \left(\lfloor\frac{n}{2}\rfloor\right)^2 + d_H^+(2, X_{12}) - \left(\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right) + \\ &\quad d_H^+(3, X_{13}) - \left(\lfloor\frac{n}{2}\rfloor\right)^2 + d_H^+(4, X_{14}) - \left(\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right) + \ldots + \\ &\quad d_H^+(n-1, X_{1n-1}) - \left(\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right) + d_H^+(n, X_{1n}) - \left(\lfloor\frac{n}{2}\rfloor\right)^2 \\ &= \left[d_H^+(1, X_{11}) + d_H^+(2, X_{12}) + \ldots + d_H^+(n, X_{1n})\right] - \\ &\quad \left[\lceil\frac{n}{2}\rceil\left(\lfloor\frac{n}{2}\rfloor\right)^2 + \lfloor\frac{n}{2}\rfloor\left(\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right)\right]. \end{aligned} \tag{2.14}$$

It may be noted that D_2^o = D_1^o. In a manner similar to (2.14) we add $d^+_{G^2}(2n+1, X_{12n+1})$, $d^+_{G^2}(2n+2, X_{12n+2})$, ... , $d^+_{G^2}(3n, X_{13n})$, and let this result be D_3^o.

$$\begin{aligned} D_3^o &= d^+_{G^2}(2n+1, X_{12n+1}) + d^+_{G^2}(2n+2, X_{12n+2}) + \ldots + d^+_{G^2}(3n, X_{13n}) \\ &= \left[d^+_H(2n+1, X_{12n+1}) + d^+_H(2n+2, X_{12n+2}) + \ldots + d^+_H(3n, X_{13n})\right] - \\ &\quad \left[\lceil\frac{n}{2}\rceil\left(\left(\lfloor\frac{n}{2}\rfloor-1\right)\lfloor\frac{n}{2}\rfloor\right) + \lfloor\frac{n}{2}\rfloor\left(\left(\lfloor\frac{n}{2}\rfloor-1\right)\lceil\frac{n}{2}\rceil\right)\right]. \end{aligned} \tag{2.15}$$

Similar to (2.14) and (2.15), for an ith-row (odd i) vertices ($5 \le i \le n-2$) of G^2, we can determine D_5^o, D_7^o, ... , D_{n-2}^o. Notice that D_4^o = D_3^o, D_6^o = D_5^o, ... , D_{n-1}^o = D_{n-2}^o and D_n^o = D_n^e. Thus adding D_1^o, D_2^o, ... , D_{n-1}^o and D_n^o, we get $W(G^2)$ as

$$\begin{aligned} W(G^2) &= \left[d^+_H(1, X_{11}) + d^+_H(2, X_{12}) + \ldots + d^+_H((n-1)n, X_{1(n-1)n}) + D_n^o\right] - \\ &\quad 2\left[\lceil\frac{n}{2}\rceil\left(\lfloor\frac{n}{2}\rfloor\right)^2 + \lfloor\frac{n}{2}\rfloor\left(\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right)\right] + \\ &\quad 2\left[\lceil\frac{n}{2}\rceil\left(\left(\lfloor\frac{n}{2}\rfloor-1\right)\lfloor\frac{n}{2}\rfloor\right) + \lfloor\frac{n}{2}\rfloor\left(\left(\lfloor\frac{n}{2}\rfloor-1\right)\lceil\frac{n}{2}\rceil\right)\right] + \ldots + \\ &\quad 2\left[2\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor\right] \\ &= W(H) - \frac{(n^2-1)^2}{8}. \end{aligned} \tag{2.16}$$

Inserting (2.10) into (2.16), we get

$$W(G^2) = n^2\left(\frac{n^3-n}{6} + \lfloor\frac{n^2}{4}\rfloor\right) - \frac{(n^2-1)^2}{8}. \tag{2.17}$$

Simplifying (2.17), we get

$$W(G^2) = \frac{1}{24}\left(4n^5 + 3n^4 - 4n^3 - 3\right). \tag{2.18}$$

Thus the result follows from (2.12) and (2.18). □

The following two results are from [7].

Lemma 2.4. *For u,v ∈ G,* $d_{G^k}(u, v) = \lceil \frac{d_G(u,v)}{k} \rceil$.

Lemma 2.5. *Let P_n be a path of order n. Then*

$$W(P_n^2) = \frac{1}{2}\left(\frac{n^3-n}{6} + \left\lfloor\frac{n^2}{4}\right\rfloor\right). \tag{2.19}$$

Lemma 2.6. *Let P_n be a path of order n and $1 \le l \le k$. Then $W(P_n^k) < W(P_n^l)$.*

Proof. Since $V(P_n^l)$ = $V(P_n^k)$ and $E(P_n^l) \subset E(P_n^k)$, implies $diam(P_n^l) > diam(P_n^k)$. Hence the bound $W(P_n^l) > W(P_n^k)$ holds. □

Finally, we give a closed form expression for $W(P_n^3)$. Let P_n be a path of order n and $1 \le l \le k$. Then

$$W(P_n^3) = W(P_n^2) - \left\lceil\frac{W(P_{n-1}^2)}{3}\right\rceil + 1. \tag{2.20}$$

The expression (2.20) is correct but we are unable to provide a proof of how such expression has resulted.

Substituting (2.19) in (2.20), we get

$$W(P_n^3) = \frac{1}{2}\left(\frac{n^3-n}{6} + \left\lfloor\frac{n^2}{4}\right\rfloor\right) - \frac{1}{2}\left\lceil\frac{1}{3}\left(\frac{(n-1)^3-(n-1)}{6} + \left\lfloor\frac{(n-1)^2}{4}\right\rfloor\right)\right\rceil + 1. \tag{2.21}$$

Simplifying the right hand side of (2.21), we get

$$W(P_n^3) = \begin{cases} \frac{n(2n-1)(n+2)}{24} - \left\lceil \frac{1}{6}\left(\frac{(n-1)^3-(n-1)}{6} + \left\lfloor \frac{(n-1)^2}{4} \right\rfloor\right) \right\rceil + 1 & \text{if } n \text{ is even,} \\ \frac{1}{2}\left(\frac{n^3-n}{6} + \left\lfloor \frac{n^2}{4} \right\rfloor\right) - \left\lceil \frac{(n(n-2)(2n+1))+3}{72} \right\rceil + 1 & \text{if } n \text{ is odd.} \end{cases} \quad (2.22)$$

3 Conclusions

Wiener index is a distance based graph invariant which is one of the most popular topological indices in mathematical chemistry. The index has been used in the characterization of various types of chemical properties, including alkanes, alkenes and arenes [2]. It has been correlated with a large number of physiochemical properties, e.g. the boiling point, refractive index, surface tension and viscosity [1].

One of the fundamental problems in computational biology is the reconstruction of the phylogeny, or evolutionary history, of a set of species or genes, typically represented as a phylogenetic tree, whose leaves are labeled by species and in which each internal node represents a speciation event whereby an ancestral species gives rise to two or more child species [10]. The authors in [3] views correlations between the problem of forming phylogenetic tree and the problem of forming a tree from a graph. One such correlation between graphs and trees, or more generally between graphs and graphs, arises in the notion of graph powers.

In this paper, for a graph G = $P_n 2 P_n$ and for a path on n vertices P_n, we have presented closed form expressions for computing the Wiener indices of G^2 and P_n^3. It should be of interest if we can obtain an expression for the Wiener index of G^k for $k \geq 3$, or for some specific class of graphs such as strongly chordal graphs to obtain an asymptotically faster algorithm for computing the Wiener index of G^k.

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Competing Interests

The authors declare that no competing interests exist.

References

[1] Dobrynin AA, Entringer R, Gutman I. Wiener index of trees: Theory and applications. Acta Appl. Math. 2001;66:211-249.

[2] Klavžar S, Gutman I. Wiener number of vertex-weighted graphs and a chemical applications. Discrete Appl. Math. 1997;80:73-81.

[3] Nishimura N, Ragde P, Thilikos DM. On graph powers for leaf-labeled trees. J. Algorithms. 2002;42:69-108.

[4] Xu M, Xu JM, Hou XM. Fault diameter of cartesian product graphs. Inf. Process. Lett. 2005;93:245-248.

[5] Xu JM. Topological structure and analysis of interconnection networks. Kluwer Academic Publishers, Dordrecht; 2001.

[6] Entringer RC, Jackson DE, Synder DA. Distance in graphs. Czechoslovak Math. J. 1976;26:283-296.

[7] An X, Wu B. The Wiener index of the kth power of a graph. Appl. Math. Lett. 2008;21 436-440.

[8] Udayakumar Reddy KR. Computing Wiener index of $(P_n \square P_n)^2$. In Proc. Int'l. Conf. Adv. Recent Tech. Comm. Comp. (ARTCom). 2010;85-88.

[9] Klav$\check{z}$ar S, Rajapakse A, Gutman I. The Szeged and the Wiener index of graphs. Appl. Math. Lett. 1996;9:45-49.

[10] Chen ZZ, Jiang T, Lin G. Computing phylogenetic roots with bounded degrees and errors. SIAM J. Comput. 2003;32:864-879.

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