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The Product and the Ratio of α – μ Random Variables and Outage, Delay-Limited and Ergodic Capacities Analysis

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Abstract

Historically, the algebra of random variables has been a discipline of interest to statisticians. On the other hand, in the last five decades special functions and numerical tools have become largely available, implying other scientists to apply results regarding this area of knowledge. In special, while studying the capacity of wireless networks, electrical and network engineers rely on evaluating the probability density function of the ratio of random variables. In the present paper, the probability distribution function of α–μ random variables is derived in terms of the H-function and used to evaluate the outage, delay-limited and ergodic capacities, generalizing earlier results in the literature. The results are evaluated by means of an original Mathematica routine and shown to be in accordance with established theoretical results.

Keywords: α – μ random variables; Mellin transform; outage capacity; delay-limited capacity; ergodic capacity

2010 Mathematics Subject Classification: 33C60; 62P30; 94A05; 94A12

1 Introduction

In applied sciences, the study of performance measures are related to functions of random variables. Civil and mechanical engineers, for example, are interested in the study of safety factors, which can be treated as the product or the ratio of two random variables in most of the cases. Electrical and network engineers, on the other hand, while studying the capacity of wireless networks, are interested

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in the outage, delay-limited and ergodic capacities of the spectrum sharing systems, which ultimately can be related to the ratio of two random variables (see, for example, [1],[2]).

In the present paper, the probability density and cummulative distribution functions of the product and the ratio of an arbitrary number of α–μ random variables [3] is derived in terms of the H-function, generalizing earlier results available in the literature. In order to numerically validate the expressions, a Mathematica routine is developed and used. It is worth noticing that α–μ random variables are generalized gamma random variables with three parameters, as will be discussed subsequently.

2 The H-Funtion

The H - function (see [4],[5] and [6]), represented as a contour complex integral which contains gamma functions in integrand, is defined as:

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{array}{cccccc}(a_1,A_1), & \ldots, & (a_n,A_n), & (a_{n+1},A_{n+1}), & \ldots, & (a_p,A_p)\\ (b_1,B_1), & \ldots, & (b_m,B_m), & (b_{m+1},B_{m+1}), & \ldots, & (b_q,B_q)\end{array}\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)} z^{-s}ds, \tag{2.1}$$

where A_j and B_j are assumed to be positive quantities and all the a_j and b_j may be complex. The contour L runs from $c-i\infty$ to $c+i\infty$ such that the poles of $\Gamma(b_j+B_js),\ j=1,\ldots,m$ lie to the left of L and the poles of $\Gamma(1-a_j-A_js), j=1,\ldots,n$ lie to the right of L.

The Mellin transform of the H-function is

$$\int_0^\infty x^{s-1}H_{p,q}^{m,n}\left[cx\,\middle|\,\begin{array}{c}(a_p,A_p)\\(b_q,B_q)\end{array}\right]dx = \frac{c^{-s}\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}. \tag{2.2}$$

Given these, one shall proceed to the study of the algebra of $\alpha-\mu$ random variables.

3 The Product and the Ratio of α-μ Random Variables

A random variable G is called an $\alpha-\mu$ random variable if its probability density function is given as [2]:

$$f_G(g) = \frac{\alpha\mu^\mu g^{\frac{\alpha}{2}\mu-1}}{2\widehat{g}^{\frac{\alpha}{2}\mu}\Gamma(\mu)}\exp\left[-\mu\frac{g^{\frac{\alpha}{2}}}{\widehat{g}}\right],\ g>0;\ \alpha,\mu,\widehat{g}>0. \tag{3.1}$$

One may notice that, in (3.1), α is a nonlinearity parameter, μ is related to the number of multipath clusters, $\widehat{g}=\sqrt[\frac{\alpha}{2}]{E(G^{\frac{\alpha}{2}})}$ where $E(.)$ denotes the expectation. It may be noticed that the $\alpha-\mu$ random variable is a generalized gamma random variable GG with 3 parameters, namely $GG(\alpha\mu/2,\mu/\widehat{g},\alpha/2)$.

In order to obtain the distributions of the product and the quotient of $\alpha-\mu$ random variables, their Mellin transform is of interest.

Theorem 1. *The Mellin transform of a $\alpha-\mu$ random variable is given as:*

$$M[f_G(g)](s) = \left(\frac{\widehat{g}}{\mu^{\frac{2}{\alpha}}}\right)^{s-1} \frac{\Gamma\left[\frac{2(s-1)+\alpha\mu}{\alpha}\right]}{\Gamma[\mu]}. \tag{3.2}$$

Proof. Consider the Mellin transform of the probability density function of a $\alpha - \mu$ distribution. By means of (3.1), one gets:

$$\begin{gathered} M[f_G(g)](s) = \int_0^\infty g^{s-1} f_G(g) dg \\ = \frac{\alpha\mu^\mu}{2\widehat{g}^{\frac{\alpha}{2}\mu}\Gamma(\mu)} \int_0^\infty g^{s-2+\frac{\alpha}{2}\mu} \exp\left[-\mu \frac{g}{\widehat{g}}^{\frac{\alpha}{2}}\right] dg. \end{gathered} \tag{3.3}$$

Consider the integral:

$$\int_0^\infty x^{a-1} e^{-bx^c} dx = \frac{\Gamma\left(\frac{a}{c}\right)}{cb^{\frac{a}{c}}},\ a, b, c > 0. \tag{3.4}$$

By means of (3.4), it is easy to express (3.3) as (3.2).

□

Corollary 1. *Consider the random variable* $X = \prod_{j=1}^{N} X_j$ *in which* X_j *are independent* $\alpha - \mu$ *random variables, i.e., generalized gamma RVs* $GG(\alpha_j\mu_j/2, \mu_j/\widehat{g}_j, \alpha_j/2)$, $j = 1, .., N$. *Then, the probability density function of* X *is given by:*

$$f_X(x) = \prod_{j=1}^{N} \frac{\mu_j^{\frac{2}{\alpha_j}}}{\widehat{g}_j \Gamma(\mu_j)} H_{0,N}^{N,0}\left[\prod_{j=1}^{N} \frac{\mu_j^{\frac{2}{\alpha_j}}}{\widehat{g}_j} x \,\middle|\, \begin{matrix} - \\ (\mu_1 - \frac{2}{\alpha_1}, \frac{2}{\alpha_1}), \quad ..., \quad (\mu_N - \frac{2}{\alpha_N}, \frac{2}{\alpha_N}) \end{matrix}\right]. \tag{3.5}$$

Proof. It is known that the Mellin transform of the distribution of the product of independent random variables is the product of the Mellin transforms of each variable [4]. Thus, the Mellin transform of the distribution of the product of N independent $\alpha - \mu$ random variables is easily given by means of (3.2) as:

$$M[f_X(x)](s) = \prod_{j=1}^{N} \left(\frac{\widehat{g}_j}{\mu_j^{\frac{2}{\alpha_j}}}\right)^{s-1} \frac{\Gamma\left[\frac{2(s-1)+\alpha_j\mu_j}{\alpha_j}\right]}{\Gamma[\mu_j]}. \tag{3.6}$$

By means of (2.1) and (3.6), the representation in (3.5) easily follows. □

Theorem 2. *Consider the random variable* $\widehat{X} = \prod_{w=1}^{N_1} Y_w \prod_{j=1}^{N_2} X_j^{-1}$ *in which* Y_w *and* X_j, $w = 1, .., N_1$ *and* $j = 1, .., N_2$, *are independent* $\alpha - \mu$ *random variables, i. e., generalized gamma RVs* $GG(\alpha_w\mu_w/2, \mu_w/\widehat{g}_w, \alpha_w/2)$, $w = 1, .., N_1$ *and* $GG(\alpha_j\mu_j/2, \mu_j/\widehat{g}_j, \alpha_j/2)$, $j = 1, .., N_2$, *respectively. Thus, the probability density function of* $\widehat{X}$ *is given by:*

$$\begin{gathered} f_{\widehat{X}}(x) = \frac{\Lambda(X,Y)}{\prod_{j=1}^{N_1}\Gamma(\mu_{Y,j})\prod_{j=1}^{N_2}\Gamma(\mu_{X,j})} \times \\ \times H_{N_2,N_1}^{N_1,N_2}\left[\Lambda(X,Y)\, x \,\middle|\, \begin{matrix} (\frac{\alpha_{X,1}(1-\mu_{X,1})-2}{\alpha_{X,1}}, \frac{2}{\alpha_{X,1}}), (\frac{\alpha_{X,2}(1-\mu_{X,2})-2}{\alpha_{X,2}}, \frac{2}{\alpha_{X,2}}), ..., (\frac{\alpha_{X,N_2}(1-\mu_{X,N_2})-2}{\alpha_{X,N_2}}, \frac{2}{\alpha_{X,N_2}}) \\ (\mu_{Y,1} - \frac{2}{\alpha_{Y,1}}, \frac{2}{\alpha_{Y,1}}), (\mu_{Y,2} - \frac{2}{\alpha_{Y,2}}, \frac{2}{\alpha_{Y,2}}), ..., (\mu_{Y,N_1} - \frac{2}{\alpha_{Y,N_1}}, \frac{2}{\alpha_{Y,N_1}}) \end{matrix}\right], \end{gathered} \tag{3.7}$$

where $\Lambda(X, Y)$ is given as:

$$\Lambda\left(X,Y\right)=\left(\prod_{j=1}^{N_1}\frac{\mu_{Y,j}^{\frac{2}{\alpha_{Y,j}}}}{\widehat{g}_{Y,j}}\right)\left(\prod_{j=1}^{N_2}\frac{\widehat{g}_{X,j}}{\mu_{X,j}^{\frac{2}{\alpha_{X,j}}}}\right). \tag{3.8}$$

One may notice that Theorem 2 reduces to Corollary 1 when $N_2=0$.

Proof. It is also known that the Mellin transform of the distribution of the quotients of $\alpha-\mu$ random variables can be easily obtained by making the substitution $s=2-s$ in (3.2) for the random variable which is in the denominator of the ratio [4]. This way, by taking the random variables in the numerator as Y and the ones in the denominator as X, (3.2) easily implies (3.7). □

Theorem 3. *Consider the random variable* $\widehat{X}=\prod_{w=1}^{N_1}Y_w\prod_{j=1}^{N_2}X_j^{-1}$ *in which* Y_w *and* X_j, $w=1,..,N_1$ *and* $j=1,..,N_2$, *are independent* $\alpha-\mu$ *random variables as defined in Theorem 2. Thus, the cummulative distribution function of* $\widehat{X}$ *is given by:*

$$F_{\widehat{X}}(x)=\frac{\Lambda(X,Y)x}{\prod_{j=1}^{N_1}\Gamma(\mu_{Y,j})\prod_{j=1}^{N_2}\Gamma(\mu_{X,j})}\times$$
$$\times H_{N_2+1,N_1+1}^{N_1,N_2+1}\left[\Lambda\left(X,Y\right)x\;\middle|\;\begin{matrix}(\frac{\alpha_{X,1}(1-\mu_{X,1})-2}{\alpha_{X,1}},\frac{2}{\alpha_{X,1}}),...,(\frac{\alpha_{X,N_2}(1-\mu_{X,N_2})-2}{\alpha_{X,N_2}},\frac{2}{\alpha_{X,N_2}}),(0,1)\\(\mu_{Y,1}-\frac{2}{\alpha_{Y,1}},\frac{2}{\alpha_{Y,1}}),...,(\mu_{Y,N_1}-\frac{2}{\alpha_{Y,N_1}},\frac{2}{\alpha_{Y,N_1}}),(-1,1)\end{matrix}\right], \tag{3.9}$$

where $\Lambda(X,Y)$ is given in (3.8). Also, one may notice that when $N_2=0$, the results from Theorem 3 can be applied to Corollary 1.

Proof. The result in (3.9) is easily demonstrated by means of the following relation for the integral of the H-function:

$$\int_0^c H_{p,q}^{m,n}\left[wz\;\middle|\;\begin{matrix}(a_1,A_1),\ldots,(a_n,A_n),(a_{n+1},A_{n+1}),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_m,B_m),(b_{m+1},B_{m+1}),\ldots,(b_q,B_q)\end{matrix}\right]dz$$
$$=cH_{p+1,q+1}^{m,n+1}\left[wc\;\middle|\;\begin{matrix}(a_1,A_1),\ldots,(a_n,A_n),(0,1),(a_{n+1},A_{n+1}),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_m,B_m),(b_{m+1},B_{m+1}),\ldots,(b_q,B_q),(-1,1)\end{matrix}\right]. \tag{3.10}$$

Thus, by means of (3.7) and (3.10), the result (3.9) is obtained. □

Being the mathematical background described, one shall proceed to the application of the ratio of α–μ random variables to the study of three capacities of spectrum sharing systems.

4 Capacity Analysis of Spectrum Sharing Systems

In the present section, a direct application of the results developed in the last section is shown. In order to numerically evaluate the H-function used, an algorithm in Wolfram Language (Mathematica) is used.

4.1 Outage Capacity

In the slow fading channel, the key event of interest is outage: this is the situation when the channel is so poor that no scheme can communicate reliably at a certain fixed data rate. The largest rate of reliable communication at a certain outage probability is called the outage capacity [7]. In the case of a spectrum sharing system with only primary (PU) and secondary (SU) users, the outage probability can be mathematically described as [2]:

$$P_{out} = Pr\left\{\frac{G_1}{G_0} < \frac{N_0(2^{R_0}-1)}{Q_{pk}}\right\}, \tag{4.1}$$

in which P_{out} is the outage probability; G_0 and G_1 are the instantaneous channel power gains from the SU transmitter to the PU receiver and SU receiver, respectively; N_0 is the power spectral density; R_0 is the transmission rate and Q_{pk} is the peak interference power constraint [2].

In the case where the instantaneous channel power gains are assumed to be α–μ random variables, the application of (3.9) is straightforward with $N_1 = 1$ and $N_2 = 1$. Thus, by means of (3.9) and (4.1), the outage probability can be given as:

$$P_{out} = \frac{\Lambda(G_0,G_1)\,N_0(2^{R_0}-1)}{\Gamma(\mu_{G_0})\,\Gamma(\mu_{G_1})\,Q_{pk}} H_{2,2}^{1,2}\left[\Lambda(G_0,G_1)\,\frac{N_0(2^{R_0}-1)}{Q_{pk}}\;\middle|\;\begin{matrix}(\frac{\alpha_{G_0}(1-\mu_{G_0})-2}{\alpha_{G_0}},\frac{2}{\alpha_{G_0}}),(0,1)\\(\mu_{G_1}-\frac{2}{\alpha_{G_1}},\frac{2}{\alpha_{G_1}}),(-1,1)\end{matrix}\right], \tag{4.2}$$

with

$$\Lambda(G_0,G_1) = \frac{\mu_{G_1}^{\frac{2}{\alpha_{G_1}}}\,\widehat{g}_{G_0}}{\widehat{g}_{G_1}\,\mu_{G_0}^{\frac{2}{\alpha_{G_0}}}}. \tag{4.3}$$

Being the outage probability expressed in terms of the H-function, one shall proceed to the evaluation of the expressions.

4.1.1 Application of the Equations Developed

In order to evaluate (4.2) and show its behavior, a set of parameters is considered.

At first, one may note that in [2] an evaluation procedure has been proposed, where the ratio $\alpha_{G_1}/\alpha_{G_0}$ was considered to be a rational number. In the present paper, on the other hand, such constraint is not present, thus the value of $\alpha_{G_1}/\alpha_{G_0}$ can be an arbitrary positive number. Figure 1 shows the behavior for the set of parameters shown in Table 1.

Table 1: Parameters Used in The Simulations

Set	α_{G_1}	μ_{G_1}	α_{G_0}	μ_{G_0}	N_0	R_0
1	$\sqrt{23}$	$\sqrt{15}$	$\sqrt{5}$	$\sqrt{3}$	0.5	1
2	$\sqrt{40}$	$\sqrt{7}$	$\sqrt{8}$	$\sqrt{5}$	0.5	1
3	$\sqrt{10}$	$\sqrt{5}$	$\sqrt{3}$	$\sqrt{2}$	0.5	1
4	$\sqrt{31}$	$\sqrt{10}$	$\sqrt{2}$	$\sqrt{1.5}$	0.5	1

The simulations in Figure 1 show good agreement with the theoretical predictions, as discussed in [2].

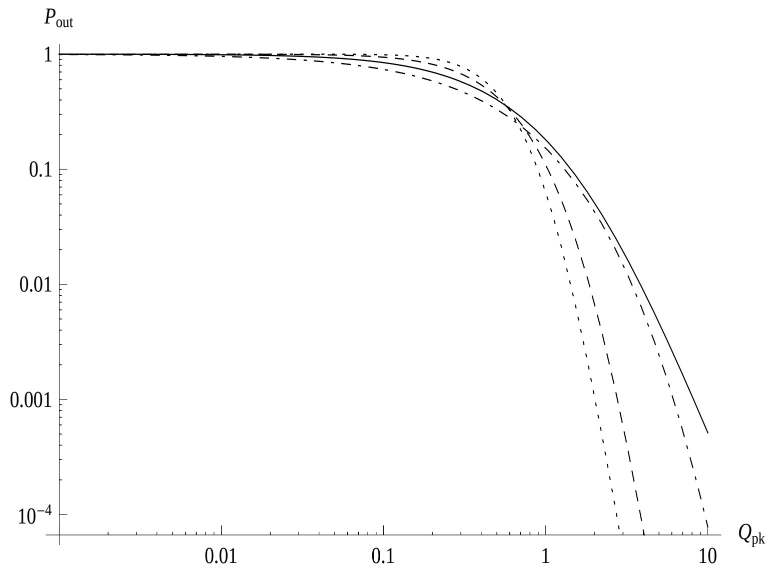

Figure 1: Behavior of the Outage Probability for Different Parameters (1 - Dashed; 2 - Dotted; 3 - Full and 4 - Dot-Dashed).

4.2 Delay-Limited Capacity

The delay-limited capacity can be defined as the maximum constant transmission rate achievable over each of the fading blocks in a spectrum sharing system [2]. Mathematically, the delay-limited capacity C_d can be defined as [2]:

$$C_d = \log_2 \left[1 + \frac{Q_{avg}}{E[G_0/G_1]N_0} \right], \tag{4.4}$$

in which Q_{avg} is the average interference power constraint [2].

In the case where the instantaneous channel power gains are assumed to be α–μ random variables, the application of (3.7) is straightforward with $N_1 = 1$ and $N_2 = 1$. Thus, by means of (3.7) and (4.4), the delay-limited capacity can be given as:

$$C_d = \log_2 \left[1 + \frac{Q_{avg}\Gamma(\mu_{G_0})\Gamma(\mu_{G_1})\mu_{G_0}^{\frac{2}{\alpha_{G_0}}}\widehat{g}_{G_1}}{N_0\Gamma\left(\mu_{G_0} + \frac{2}{\alpha_{G_0}}\right)\Gamma\left(\mu_{G_1} - \frac{2}{\alpha_{G_1}}\right)\widehat{g}_{G_0}\mu_{G_1}^{\frac{2}{\alpha_{G_1}}}} \right]. \tag{4.5}$$

The expression (4.5) has been obtained by noticing that the expectation of a given random variable is its Mellin transform with $s = 2$. Thus, by means of (2.2) and (3.7), (4.5) easily follows. Since (4.5) is expressed in terms of gamma functions and the latter are widely used in engineering, the results will not be numerically evaluated.

4.3 Ergodic Capacity

The ergodic capacity can be defined as the maximum achievable rate averaged over all the fading blocks (long-term average) in a spectrum sharing system [2]. Mathematically, the ergodic capacity $C_e^{(a)}$ can be defined as [2]:

$$C_e^{(a)} = \int_{\frac{1}{\gamma_0}}^{\infty} B \log_2(\gamma_0 x) f_{\frac{G_1}{G_0}}(x)dx, \tag{4.6}$$

in which B is the total available bandwidth, $\gamma_0 = 1/(\psi_0 N_0 B)$ and ψ_0 is calculated so that the average interference power in (4.6) equals Q_{avg} [2].

Again, in the case where the instantaneous channel power gains are assumed to be α–μ random variables, the application of (3.7) is straightforward with $N_1 = 1$ and $N_2 = 1$. Thus, by means of (3.7) and (4.6), the ergodic capacity can be given as:

$$\begin{aligned}
C_e^{(a)} &= B\int_{\frac{1}{\gamma_0}}^{\infty} \log_2(\gamma_0 x) f_{\frac{G_1}{G_0}}(x)dx \\
&= \frac{\Lambda(G_0,G_1)B}{\Gamma(\mu_{G_0})\Gamma(\mu_{G_1})}\int_{\frac{1}{\gamma_0}}^{\infty} \log_2(\gamma_0 x) H_{1,1}^{1,1}\left[\Lambda(G_0,G_1)x \,\middle|\, \begin{matrix} (\frac{\alpha_{G_0}(1-\mu_{G_0})-2}{\alpha_{G_0}}, \frac{2}{\alpha_{G_0}}) \\ (\mu_{G_1}-\frac{2}{\alpha_{G_1}}, \frac{2}{\alpha_{G_1}}) \end{matrix}\right] dx \\
&= \frac{\Lambda(G_0,G_1)B}{\Gamma(\mu_{G_0})\Gamma(\mu_{G_1})}\frac{1}{2\pi i}\int_L \Lambda(G_0,G_1)^{-s}\,\Gamma\left(\mu_{G_1}-\frac{2(1-s)}{\alpha_{G_1}}\right)\Gamma\left(\mu_{G_0}-\frac{2(s-1)}{\alpha_{G_0}}\right)\int_{\frac{1}{\gamma_0}}^{\infty}\log_2(\gamma_0 x)\,x^{-s}dxds \\
&= \frac{\Lambda(G_0,G_1)B}{\Gamma(\mu_{G_0})\Gamma(\mu_{G_1})ln(2)\gamma_0}\frac{1}{2\pi i}\int_L (\Lambda(G_0,G_1)/\gamma_0)^{-s}\,\frac{\Gamma\left(\mu_{G_1}-\frac{2(1-s)}{\alpha_{G_1}}\right)\Gamma\left(\mu_{G_0}-\frac{2(s-1)}{\alpha_{G_0}}\right)\Gamma(s-1)^2}{\Gamma(s)^2}ds \\
&= \frac{\Lambda(G_0,G_1)B}{\Gamma(\mu_{G_0})\Gamma(\mu_{G_1})ln(2)\gamma_0}H_{3,1}^{3,1}\left[\Lambda(G_0,G_1)/\gamma_0 \,\middle|\, \begin{matrix} (\frac{\alpha_{G_0}(1-\mu_{G_0})-2}{\alpha_{G_0}}, \frac{2}{\alpha_{G_0}}),(0,1),(0,1) \\ (\mu_{G_1}-\frac{2}{\alpha_{G_1}}, \frac{2}{\alpha_{G_1}}),(-1,1),(-1,1) \end{matrix}\right].
\end{aligned} \tag{4.7}$$

The result in (4.7) generalizes an earlier result [2] derived as a series for rational values of the parameters.

Since the ergodic capacity is given in terms of the H-function, it is of interest to numerically evaluate the expressions obtained.

4.3.1 Application of the Equations Developed

The same set of data used in the simulation of outage capacity and presented in Table 1 is used in the evaluation of the ergodic capacity. For every case below, $B = 1$ bps. Also, the graphics plotted are the ergodic capacity in $bits/s/Hz$ versus the parameter γ_0, differently from [2] in which the plotted graphics were the ergodic capacity in $bits/s/Hz$ versus Q_{avg}. Figure 2 decipts a plot of the results.

5 Conclusions

The algebra of random variables has shown to be a discipline of interest not only to statisticians, but also to applied scientists. In special, the capacity evaluation of spectrum sharing systems is of great interest to electrical and network engineers.

In the present paper the probability density function of the product and the ratio of an arbitrary number of α–μ random variables has been derived in terms of the H-function. In order to evaluate the expressions hereby developed, a Mathematica code has been mounted and used.

Numerical simulations of the outage probability have been carried out. The simulations performed show good agreement to the theoretical results and generalize the ones previously given in the literature. In addition, delay-limited and ergodic capacities have been obtained and expressed in terms of the H-function in a general set-up.

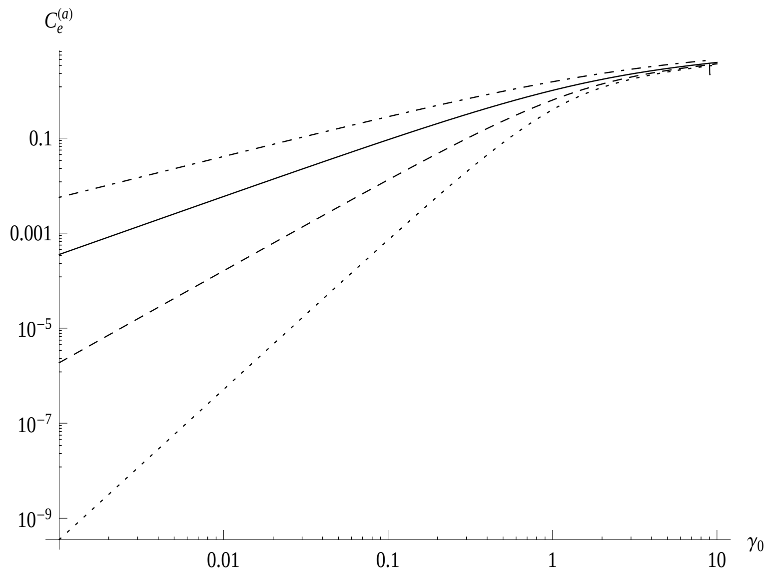

Figure 2: Behavior of the ergodic Capacity for Different Parameters (1 - Dashed; 2 - Dotted; 3 - Full and 4 - Dot-Dashed).

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Competing Interests

The authors declare that no competing interests exist.

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