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Intuitionistic fuzzy subgroups with respect to norms (T, S)

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Abstract: The purpose of this paper is introduce the notion of intuitionistic fuzzy subgroups with respect to norms (t-norm T and s-norm S). Also we introduce intersection and normality of them and investigate some properties of them. Finally, we provide some results of them under group homomorphisms.

Keywords: Group theory, theory of fuzzy sets, intuitionistic fuzzy groups, norms, homomorphisms, intersection.

1. Introduction

Group theory, in modern algebra, the study of groups, which are systems consisting of a set of elements and a binary operation that can be applied to two elements of the set, which together satisfy certain axioms. Groups are vital to modern algebra; their basic structure can be found in many mathematical phenomena. Groups can be found in geometry, representing phenomena such as symmetry and certain types of transformations. Group theory has applications in physics, chemistry, and computer science, and even puzzles like Rubik's Cube can be represented using group theory.

In 1965, Zadeh [1] introduced the notion of fuzzy sets. In 1971, Rosenfled [2] introduced fuzzy sets in the realm of group theory and formulated the concepts of fuzzy subgroups of a group. An increasing number of properties from classical group theory have been generalized. Many authors have worked on fuzzy group theory [3–5]. Especially, some authors considered the fuzzy subgroups with respect to a t-norm and gave some results [5–7]. The concept of intuitionistic fuzzy set was introduced by Atanassov [8], as a generalization of the notion of fuzzy set. The theory of intuitionistic fuzzy set is expected to play an important role in modern mathematics in general as it represents a generalization of fuzzy set. After the concept of intuitionistic fuzzy set was introduced, several papers have been published by mathematicians to extend the classical mathematical concepts and fuzzy mathematical concepts to the case of intuitionistic fuzzy mathematics. In the case of intuitionistic fuzzy mathematics, there were some attempts to establish a significant and rational definition of intuitionistic fuzzy group. Zhan and Tan [9] defined intuitionistic fuzzy subgroup as a generalization of Rosenfeld's fuzzy subgroup.

By starting with a given classical group they define intuitionistic fuzzy subgroup using the classical binary operation defined over the given classical group. The author by using norms, investigated some properties of fuzzy algebraic structures [10]. In this paper, by using norms(t-norm T and s-norm S) we define intuitionistic fuzzy subgroups of group G as $IFGN(G)$ and normality of G as $NIFGN(G)$. Also we investigate algebriac structures and some related properties of them and prove that if $A, B \in IFGN(G)$ and $A, B \in NIFGN(G)$, then $A \cap B \in IFGN(G)$ and $A \cap B \in NIFGN(G)$. Next we define normality between $A, B \in IFGN(G)$ as $A \blacktriangleright B$ and give characterizations about them. Later we investigate them under group homomorphism $\varphi : G \to H$ such that if $A \in IFGN(G)$ and $B \in IFGN(H)$, then $\varphi(A) \in IFGN(H)$ and $\varphi^{-1}(B) \in IFGN(G)$. Also if $A \in NIFGN(G)$ and $B \in NIFGN(H)$, then $\varphi(A) \in NIFGN(H)$ and $\varphi^{-1}(B) \in NIFGN(G)$. Finally, if $A, B \in IFGN(G)$ and $A \blacktriangleright B$, then $\varphi(A) \blacktriangleright \varphi(B)$ and if $A, B \in IFGN(H)$ and $A \blacktriangleright B$, then $\varphi^{-1}(A) \blacktriangleright \varphi^{-1}(B)$.

2. preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequel. For details we refer to [6,8,11–16].

Definition 1. A group is a non-empty set G on which there is a binary operation $(a, b) \to ab$ such that

(1) if a and b belong to G then ab is also in G (closure),
(2) $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
(3) there is an element $e_G \in G$ such that $ae_G = ee_G a = a$ for all $a \in G$ (identity),
(4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e_G$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, *i.e.*, $ab = ba$ for all $a, b \in G$.

Remark 1. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e.

Proposition 1. *Let G be a group. Let H be a non-empty subset of G. The following are equivalent:*

(1) H is a subgroup of G.
(2) $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

Definition 2. Let H be subgroup of group G. Then we say that H is normal subgroup of G if for all $g \in G$ and $h \in H$, we have that $ghg^{-1} \in H$.

Definition 3. Let G and H be any two groups and $f : G \to H$ be a function. Then f is called a homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in G$.

Definition 4. Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0, 1]$.

Definition 5. For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 6. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$, respectively.

We will denote the set of all $IFSs$ in X as $IFS(X)$.

Definition 7. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.

Definition 8. A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:

(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),

for all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 1. (1) Standard intersection T-norm $T_m(x, y) = \min\{x, y\}$.
(2) Bounded sum T-norm $T_b(x, y) = \max\{0, x + y - 1\}$.
(3) algebraic product T-norm $T_p(x, y) = xy$.

(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \le T(x,y) \le T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Recall that t-norm T will be idempotent if for all $x \in [0,1]$ we have $T(x,x) = x$.

Lemma 1. *Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 9. An s-norm S is a function $S : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(1) $S(x,0) = x$,
(2) $S(x,y) \le S(x,z)$ if $y \le z$,
(3) $S(x,y) = S(y,x)$,
(4) $S(x, S(y,z)) = S(S(x,y),z)$,

for all $x, y, z \in [0,1]$.

We say that S is idempotent if for all $x \in [0,1]$, $S(x,x) = x$.

Example 2. The basic S-norms are $S_m(x,y) = \max\{x,y\}$, $S_b(x,y) = \min\{1, x+y\}$ and $S_p(x,y) = x + y - xy$ for all $x, y \in [0,1]$, where S_m is standard union, S_b is bounded sum and S_p is algebraic sum.

Lemma 2. *Let S be a s-norm. Then $S(S(x,y), S(w,z)) = S(S(x,w), S(y,z))$, for all $x, y, w, z \in [0,1]$.*

Definition 10. Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(X)$. Define intesection A and B as

$$A \cap B = (\mu_A, \nu_A) \cap (\mu_B, \nu_B) = (\mu_{A\cap B}, \nu_{A\cap B})$$

such that $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = S(\nu_A(x), \nu_B(y))$ for all $x \in X$.

Definition 11. Let φ be a function from set X into set Y such that $A = (\mu_A, v_A)$ and $B = (\mu_B, v_B)$ be two intuitionistic fuzzy sets in X and Y respectively. For all $x \in X$ and $y \in Y$, we define

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y))$$

$$= \begin{cases} (\sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset; \\ (0,1) & \text{if } \varphi^{-1}(y) = \emptyset. \end{cases}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

3. Intuitionistic fuzzy subgroups with respect to norms (t-norm T and s-norm S)

Definition 12. Let G be a group. An $A = (\mu_A, \nu_A) \in IFS(G)$ is said to be intuitionistic fuzzy subgroup with respect to norms(t-norm T and s-norm S) (in short, $IFGN(G)$) of G if

(1) $A(xy) \supseteq (T(\mu_A(x), \mu_A(y)), S(\nu_A(x), \nu_A(y)))$,
(2) $A(x^{-1}) \supseteq A(x)$,

for all $x, y \in G$.

Remark 2. Conditions (1) and (2) of Definition 12 are equivalent to following conditions:

(1) $\mu_A(xy) \geq T(\mu_A(x), \mu_A(y))$,
(2) $\mu_A(x^{-1}) \geq \mu_A(x)$,
(3) $\nu_A(xy) \leq S(\nu_A(x), \nu_A(y))$,
(4) $\nu_A(x^{-1}) \leq \nu_A(x)$,

for all $x, y \in G$.

Example 3. Let $(\mathbb{Z}, +)$ be a group of integers. For all $x \in G$ we define a fuzzy subset μ_A and ν_A of G as

$$\mu_A(x) = \begin{cases} 0.65 & \text{if } x \in \{0, \pm 2, \pm 4, ...\}; \\ 0.35 & \text{if } x \in \{\pm 1, \pm 3, ...\}. \end{cases}$$

$$\nu_A(x) = \begin{cases} 0.20 & \text{if } x \in \{0, \pm 2, \pm 4, ...\}; \\ 0.80 & \text{if } x \in \{\pm 1, \pm 3, ...\}. \end{cases}$$

Let $T(x, y) = T_p(x, y) = xy$ and $S(x, y) = S_p(x, y) = x + y - xy$ for all $x, y \in [0, 1]$, then $A = (\mu_A, \nu_A) \in IFGN(G)$.

Lemma 3. *Let $A = (\mu_A, \nu_A) \in IFS(G)$ such that G is finite group and T and S be idempotent. If $A = (\mu_A, \nu_A)$ satisfies condition (1) of Definition 12, then $A = (\mu_A, \nu_A) \in IFGN(G)$.*

Proof. As G is finite so we have an $x \in G$ such that $x \neq e$ and x has finite order, say $n > 1$ then $x^n = e$ and $x^{-1} = x^{n-1}$. Now by using condition (1) of Definition 12 repeatedly, we get that

$$\mu_A(x^{-1}) = \mu(x^{n-1}) = \mu_A(x^{n-2}x) \geq T(\mu_A(x^{n-1}), \mu_A(x)) \geq T(\underbrace{\mu_A(x), \mu_A(x), ..., \mu_A(x)}_{n}) = \mu_A(x)$$

and

$$\nu_A(x^{-1}) = \nu_A(x^{n-1}) = \nu_A(x^{n-2}x) \leq (\nu_A(x^{n-1}), \nu_A(x)) \leq S(\underbrace{\nu_A(x), \nu_A(x), ..., \nu_A(x)}_{n}) = \nu_A(x).$$

Thus

$$A(x^{-1}) = (\mu_A(x^{-1}), \nu_A(x^{-1})) \supseteq (\mu_A(x), \nu_A(x)) = A(x).$$

Hence $A = (\mu_A, \nu_A) \in IFGN(G)$. □

Proposition 2. *Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and T and S be idempotent. Then for all $x \in G$, and $n \geq 1$,*

(1) $A(e) \supseteq A(x)$;
(2) $A(x^n) \supseteq A(x)$;
(3) $A(x) = A(x^{-1})$.

Proof. Let $x \in G$ and $n \geq 1$.

(1)

$$\mu_A(e) = \mu_A(xx^{-1}) \geq T(\mu_A(x), \mu_A(x^{-1})) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)$$

and
$$\nu_A(e) = \nu_A(xx^{-1}) \leq S(\nu_A(x), \nu_A(x^{-1})) \leq S(\nu_A(x), \nu_A(x)) = \nu_A(x).$$

Hence
$$A(e) = (\mu_A(e), \nu_A(e)) \supseteq (\mu_A(x), \nu_A(x)) = A(x).$$

(2)
$$\mu_A(x^n) = \mu_A(\underbrace{xx...x}_{n}) \geq T(\underbrace{\mu_A(x), \mu_A(x), ..., \mu_A(x)}_{n}) = \mu_A(x)$$

and
$$\nu_A(x^n) = \nu_A(\underbrace{xx...x}_{n}) \leq S(\underbrace{\nu_A(x), \nu_A(x), ..., \nu_A(x)}_{n}) = \nu_A(x).$$

Hence
$$A(x^n) = (\mu_A(x^n), \nu_A(x^n)) \supseteq (\mu_A(x), \nu_A(x)) = A(x).$$

(3) As
$$\mu_A(x) = \mu_A((x^{-1}))^{-1} \geq \mu_A(x^{-1}) \geq \mu_A(x)$$

and
$$\nu_A(x) = \nu_A((x^{-1}))^{-1} \leq \nu_A(x^{-1}) \leq \nu_A(x) = \nu_A(x^{-1}).$$

So $\mu_A(x) = \mu_A(x^{-1})$ and $\nu_A(x) = \nu_A(x^{-1})$, therefore
$$A(x) = (\mu_A(x), \nu_A(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1})) = A(x^{-1}).$$

□

Proposition 3. *Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and T and S be idempotent. Then $A(xy) = A(y)$ if and only if $A(x) = A(e)$ for all $x, y \in G$.*

Proof. Let $A(xy) = A(y)$ for all $x, y \in G$. If we let $y = e$, then we get that $A(x) = A(e)$.

Conversely, suppose that $A(x) = A(e)$ so from Proposition 2(1), we get that $A(x) \supseteq A(y)$ and $A(x) \supseteq A(xy)$ which mean that $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(x) \geq \mu_A(xy)$ and $\nu_A(x) \leq \nu_A(y)$ and $\nu_A(x) \leq \nu_A(xy)$. Then
$$\begin{aligned}
\mu_A(xy) &\geq T(\mu_A(x), \mu_A(y)) \\
&\geq T(\mu_A(y), \mu_A(y)) \\
&= \mu_A(y) \\
&= \mu_A(x^{-1}xy) \\
&\geq T(\mu_A(x), \mu_A(xy)) \\
&\geq T(\mu_A(xy), \mu_A(xy)) = \mu_A(xy).
\end{aligned}$$

So
$$\mu_A(xy) = \mu_A(y). \tag{1}$$

Also
$$\begin{aligned}
\nu_A(xy) &\leq S(\nu_A(x), \nu_A(y)) \\
&\leq S(\nu_A(y), \nu_A(y)) \\
&= \nu_A(y) \\
&= \nu_A(x^{-1}xy) \\
&\leq S(\nu_A(x), \nu_A(xy)) \\
&\leq S(\nu_A(xy), \nu_A(xy)) \\
&= \nu_A(xy).
\end{aligned}$$

So
$$\nu_A(xy) = \nu_A(y). \tag{2}$$

Hence
$$A(xy) = (\mu_A(xy), \nu_A(xy)) = (\mu_A(y), \nu_A(y)) = A(y).$$

$\square$

Definition 13. Let $A = (\mu_A, \nu_A) \in IFS(G)$ and $B = (\mu_B, \nu_B) \in IFS(G)$. We define the composion of A and B as $A \circ B \in IFS(G)$ such that for all $x \in G$, we have
$$(A \circ B)(x) = ((\mu_A, \nu_A) \circ (\mu_B, \nu_B))(x) = (\mu_{A\circ B}(x), \nu_{A\circ B}(x))$$

such that
$$\mu_{A\circ B}(x) = \begin{cases} \sup_{x=yz} T((\mu_A(y), \mu_A(z)) & \text{if } x = yz; \\ 0 & \text{if } x \neq yz, \end{cases}$$

and
$$\nu_{A\circ B}(x) = \begin{cases} \inf_{x=yz} S((\nu_A(y), \nu_A(z)) & \text{if } x = yz; \\ 0 & \text{if } x \neq yz. \end{cases}$$

Proposition 4. *Let $A^{-1} = (\mu_A^{-1}, \nu_A^{-1}) \in IFS(G)$ be the inverse of $A = (\mu_A, \nu_A) \in IFS(G)$ such that for all $x \in G$*
$$A^{-1}(x) = (\mu_A^{-1}(x), \nu_A^{-1}(x)) = (\mu_A(x^{-1}), \nu_A(x^{-1})) = A(x^{-1}).$$

Let T and S be idempotent then $A \in IFGN(G)$ if and only if A satisfies the following conditions:

(1) $A \circ A \subseteq A$;
(2) $A^{-1} = A$.

Proof. Let $x, y, z \in G$ such that $x = yz$. If $A = (\mu_A, \nu_A) \in IFGN(G)$, then
$$\mu_A(x) = \mu_A(yz) \geq T(\mu_A(y), \mu_A(z)) = \mu_{A\circ A}(x)$$

and
$$\nu_A(x) = \nu_A(yz) \leq S(\nu_A(y), \nu_A(z)) = \nu_{A\circ A}(x).$$

Which yield
$$(A \circ A)(x) = (\mu_{A\circ A}(x), \nu_{A\circ A}(x)) \subseteq (\mu_A(x), \nu_A(x)) = A(x).$$

Then $A \circ A \subseteq A$. Also $A^{-1} = A$ comes from Proposition 2(3).
Conversely, let $A \circ A \subseteq A$ and $A^{-1} = A$. As $A \circ A \subseteq A$, so
$$\mu_A(yz) = \mu_A(x) \geq \mu_{A\circ A}(x) = \sup_{x=yz} T(\mu_A(y), \mu_A(z)) \geq T(\mu_A(y), \mu_A(z))$$

and
$$\nu_A(yz) = \nu_A(x) \leq \nu_{A\circ A}(x) = \inf_{x=yz} S(\nu_A(y), \nu_A(z)) \leq S(\nu_A(y), \nu_A(z)).$$

Which mean that
$$A(yz) = (\mu_A(yz), \nu_A(yz)) \supseteq (T(\mu_A(y), \mu_A(z)), S(\nu_A(y), \nu_A(z))). \tag{3}$$

As $A^{-1} = A$, so
$$(x) = (\mu_A(x), \nu_A(x)) = (\mu_A^{-1}(x), \nu_A^{-1}(x)) = A^{-1}(x). \tag{4}$$

Therefore from (3) and (4) we get that $A \in IFGN(G)$. $\square$

Corollary 1. *Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and $B = (\mu_B, \nu_B) \in IFGN(G)$ and G be commutative group. Then $A \circ B \in IFGN(G)$ if and only if $A \circ B = B \circ A$.*

Proof. If $A, B, A \circ B \in IFGN(G)$, then from Proposition 4 we get that $A^{-1} = A, B^{-1} = B$ and $(B \circ A)^{-1} = B \circ A$. Now $A \circ B = A^{-1} \circ B^{-1} = (B \circ A)^{-1} = B \circ A$.

Conversely, since $A \circ B = B \circ A$ we have

$$(A \circ B)^{-1} = (B \circ A)^{-1} = A^{-1} \circ B^{-1} = A \circ B.$$

Also

$$(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B = A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B) \subseteq B \circ B.$$

Now Proposition 4 gives us that $A \circ B \in IFGN(G)$. $\square$

Proposition 5. *Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and $B = (\mu_B, \nu_B) \in IFGN(G)$. Then $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in IFGN(G)$.*

Proof. Let $x, y \in G$. Then

$$\begin{aligned} \mu_{A\cap B}(xy) &= T(\mu_A(xy), \mu_B(xy)) \\ &\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\ &= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \\ &= T(\mu_{A\cap B}(x), \mu_{A\cap B}(y)). \end{aligned}$$

And

$$\begin{aligned} \nu_{A\cap B}(xy) &= S(\nu_A(xy), \nu_B(xy)) \\ &\leq S(S(\nu_A(x), \nu_A(y)), S(\nu_B(x), \nu_B(y))) \\ &= S(S(\nu_A(x), \nu_B(x)), S(\nu_A(y), \nu_B(y))) \\ &= S(\nu_{A\cap B}(x), \nu_{A\cap B}(y)). \end{aligned}$$

Which mean that

$$(A \cap B)(xy) = (\mu_{A\cap B}(xy), \nu_{A\cap B}(xy)) \supseteq (T(\mu_{A\cap B}(x), \mu_{A\cap B}(y)), S(\nu_{A\cap B}(x), \nu_{A\cap B}(y))).$$

Also

$$\mu_{A\cap B}(x^{-1}) = T(\mu_A(x^{-1}), \mu_B(x^{-1})) \geq T(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x)$$

and

$$\nu_{A\cap B}(x^{-1}) = S(\nu_A(x^{-1}), \nu_B(x^{-1})) \leq S(\nu_A(x), \nu_B(x)) = \nu_{A\cap B}(x).$$

So

$$(A \cap B)(x^{-1}) = (\mu_{A\cap B}(x^{-1}), \nu_{A\cap B}(x^{-1})) \supseteq (\mu_{A\cap B}(x), \nu_{A\cap B}(x)) = (A \cap B)(x).$$

Thus $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in IFGN(G)$. $\square$

Corollary 2. *Let $I_n = \{1, 2, ..., n\}$. If $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq IFGN(G)$. Then $A = \cap_{i\in I_n} A_i \in IFGN(G)$.*

Definition 14. We say that $A = (\mu_A, \nu_A) \in IFGN(G)$ is normal if for all $x, y \in G$, $A(xyx^{-1}) = A(y)$. Also we denote by $NIFGN(G)$ the set of all normal intuitionistic fuzzy groups with respect to norms (t-norm T and s-norm S).

Proposition 6. *Let $A = (\mu_A, \nu_A) \in NIFGN(G)$ and $B = (\mu_B, \nu_B) \in NIFGN(G)$. Then $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in NIFGN(G)$.*

Proof. As Proposition 5 we have that $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in IFGN(G)$. Let $x, y, \in G$, then

$$\mu_{A\cap B}(xyx^{-1}) = T(\mu_A(xyx^{-1}), \mu_B(xyx^{-1})) = T(\mu_A(y), \mu_B(y)) = \mu_{A\cap B}(y)$$

and

$$\nu_{A\cap B}(xyx^{-1}) = S(\nu_A(xyx^{-1}), \nu_B(xyx^{-1})) = S(\nu_A(y), \nu_B(y)) = \nu_{A\cap B}(y).$$

Thus

$$(A \cap B)(xyx^{-1}) = (\mu_{A\cap B}(xyx^{-1}), \nu_{A\cap B}(xyx^{-1})) = (\mu_{A\cap B}(y), \nu_{A\cap B}(y)) = (A \cap B)(y).$$

Therefore $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in NIFGN(G)$. $\square$

Corollary 3. *Let $I_n = \{1, 2, ..., n\}$. If $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq NIFGN(G)$. Then $A = \cap_{i\in I_n}\mu_i \in NIFGN(G)$.*

Definition 15. Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and $B = (\mu_B, \nu_B) \in IFGN(G)$ such that $A \subseteq B$. Then A is called normal of B, written $A \blacktriangleright B$, if for all $x, y \in G$ we have

$$A(xyx^{-1}) = (\mu_A(xyx^{-1}), \nu_A(xyx^{-1})) \supseteq (T(\mu_A(y), \mu_B(x)), S(\nu_A(y), \nu_B(x))).$$

Proposition 7. *(1) Let G_1 and G_2 are subgroups of G. Then G_1 is a normal subgroup of G_2 if and only if $1_{G_1} \blacktriangleright 1_{G_2}$.*
(2) If T and S be idempotent, then every intuitionistic fuzzy subgroup with respect to norms(t-norm T and s-norm S) is normal fuzzy subgroup of itself.

Proof. (1) Let $x \in G_2$ and $y \in G_1$ then $1_{G_2}(x) = 1$ and $1_{G_1}(y) = 1$. If $G_1 \trianglerighteq G_2$, then $xyx^{-1} \in G_1$ and so $1_{G_1}(xyx^{-1}) = 1$. As $1_G = (1_G, 1_G) \in IFGN(G)$, so

$$1_{G_1}(xyx^{-1}) = 1 \geq 1 = T(1, 1) = T(1_{G_1}(y), 1_{G_2}(x))$$

and

$$1_{G_1}(xyx^{-1}) = 1 \leq 1 = S(1, 1) = S(1_{G_1}(y), 1_{G_2}(x)).$$

Then

$$1_{G_1}(xyx^{-1}) = (1_{G_1}(xyx^{-1}), 1_{G_1}(xyx^{-1})) \supseteq (T(1_{G_1}(y), 1_{G_2}(x)), S(1_{G_1}(y), 1_{G_2}(x))).$$

Hence $1_{G_1} \blacktriangleright 1_{G_2}$.
(2) Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and $x, y \in G$ then

$$\begin{aligned}\mu_A(xyx^{-1}) &\geq T(\mu_A(xy), \mu_A(x^{-1}))\\ &\geq T(T(\mu_A(x), \mu_A(y)), \mu_A(x))\\ &= T(T(\mu_A(y), \mu_A(x)), \mu_A(x))\\ &= T(\mu_A(y), T(\mu_A(x), \mu_A(x)))\\ &= T(\mu_A(y), \mu_A(x)).\end{aligned}$$

And

$$\begin{aligned}\nu_A(xyx^{-1}) &\leq S(\nu_A(xy), \nu_A(x^{-1}))\\ &\leq S(S(\nu_A(x), \nu_A(y)), \nu_A(x))\\ &= S(S(\nu_A(y), \nu_A(x)), \nu_A(x))\\ &= S(\nu_A(y), S(\nu_A(x), \nu_A(x)))\\ &= S(\nu_A(y), \nu_A(x)).\end{aligned}$$

Thus

$$A(xyx^{-1}) = (\mu_A(xyx^{-1}), \nu_A(xyx^{-1})) \supseteq (T(\mu_A(y), \mu_A(x)), S(\nu_A(y), \nu_B(x))).$$

Hence $A = (\mu_A, \nu_A) \blacktriangleright A = (\mu_A, \nu_A)$.
□

Proposition 8. *Let $A = (\mu_A, \nu_A) \in NIFGN(G)$ and $B = (\mu_B, \nu_B) \in IFGN(G)$ such that T and S be idempotent. Then $A \cap B \blacktriangleright B$.*

Proof. Using Proposition 5, we get that $A \cap B \in IFGN(G)$. Let $x, y \in G$, then

$$\begin{aligned}
\mu_{A\cap B}(xyx^{-1}) &= T(\mu_A(xyx^{-1}), \mu_B(xyx^{-1})) \\
&= T(\mu_A(y), \mu_B(xyx^{-1})) \\
&\geq T(\mu_A(y), T(\mu_B(xy), \mu_B(x^{-1})) \\
&\geq T(\mu_A(y), T(T(\mu_B(x), \mu_B(y)), \mu_B(x))) \\
&= T(\mu_A(y), T(\mu_B(y), T(\mu_B(x), \mu_B(x)))) \\
&= T(\mu_A(y), T(\mu_B(y), \mu_B(x))) \\
&= T(T(\mu_A(y), \mu_B(y)), \mu_B(x)) \\
&= T(\mu_{A\cap B}(y), \mu_B(x)).
\end{aligned}$$

And

$$\begin{aligned}
\nu_{A\cap B}(xyx^{-1}) &= S(\nu_A(xyx^{-1}), \nu_B(xyx^{-1})) \\
&= S(\nu_A(y), \nu_B(xyx^{-1})) \\
&\leq S(\nu_A(y), S(\nu_B(xy), \nu_B(x^{-1})) \\
&\leq S(\nu_A(y), S(S(\nu_B(x), \nu_B(y)), \nu_B(x))) \\
&= S(\nu_A(y), S(\nu_B(y), S(\nu_B(x), \nu_B(x)))) \\
&= S(\nu_A(y), S(\nu_B(y), \nu_B(x))) \\
&= S(S(\nu_A(y), \nu_B(y)), \nu_B(x)) \\
&= S(\nu_{A\cap B}(y), \nu_B(x)).
\end{aligned}$$

Thus

$$(A \cap B)(xyx^{-1}) = (\mu_{A\cap B}(xyx^{-1}), \nu_{A\cap B}(xyx^{-1})) \supseteq (T(\mu_{A\cap B}(y), \mu_B(x)), S(\nu_{A\cap B}(y), \nu_B(x))).$$

Which means that $A \cap B \blacktriangleright B$. □

Proposition 9. *Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and $B = (\mu_B, \nu_B) \in IFGN(G)$ and $C = (\mu_C, \nu_C) \in IFGN(G)$ such that T and S be idempotent. If $A \blacktriangleright C$ and $B \blacktriangleright C$, then $A \cap B \blacktriangleright C$.*

Proof. By Proposition 5 we will have that $A \cap B \in IFGN(G)$. Let $x, y \in G$, then

$$\begin{aligned}
\mu_{A\cap B}(xyx^{-1}) &= T(\mu_A(xyx^{-1}), \mu_B(xyx^{-1})) \\
&\geq T(T(\mu_A(y), \mu_C(x)), T(\mu_B(y), \mu_C(x))) \\
&= T(T(\mu_A(y), \mu_B(y)), T(\mu_C(x), \mu_C(x))) \\
&= T(T(\mu_A(y), \mu_B(y)), \mu_C(x)) = T(\mu_{A\cap B}(y), \mu_C(x)).
\end{aligned}$$

And

$$\begin{aligned}
\nu_{A\cap B}(xyx^{-1}) &= S(\nu_A(xyx^{-1}), \nu_B(xyx^{-1})) \\
&\leq S(S(\nu_A(y), \nu_C(x)), S(\nu_B(y), \nu_C(x))) \\
&= S(S(\nu_A(y), \nu_B(y)), S(\nu_C(x), \nu_C(x))) \\
&= S(S(\nu_A(y), \nu_B(y)), \nu_C(x)) = S(\nu_{A\cap B}(y), \nu_C(x)).
\end{aligned}$$

Therefore

$$(A \cap B)(xyx^{-1}) = (\mu_{A\cap B}(xyx^{-1}), \nu_{A\cap B}(xyx^{-1})) \supseteq (T(\mu_{A\cap B}(y), \mu_C(x)), S(\nu_{A\cap B}(y), \nu_C(x))).$$

Hence $A \cap B \blacktriangleright C$. $\square$

Corollary 4. *Let $I_n = \{1, 2, ..., n\}$ and $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \subseteq IFGN(G)$ such that $\{A_i = (\mu_{A_i}, \nu_{A_i}) \mid i \in I_n\} \blacktriangleright B = (\mu_B, \nu_B)$. Then $A = \cap_{i\in I_n} A_i \blacktriangleright B = (\mu_B, \nu_B)$.*

4. Homomorphisms of $IFGN(G)$

Proposition 10. *Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and H be a group. Suppose that $\varphi : G \to H$ is a homomorphism. Then $\varphi(A) \in IFGN(H)$.*

Proof. Let $u, v \in H$ and $x, y \in G$ such that $u = \varphi(x)$ and $v = \varphi(y)$ and $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A))$. Now

$$\begin{aligned}
\varphi(\mu_A)(uv) &= \sup\{\mu_A(xy) \mid u = \varphi(x), v = \varphi(y)\} \\
&\geq \sup\{T(\mu_A(x), \mu_A(y)) \mid u = \varphi(x), v = \varphi(y)\} \\
&= T(\sup\{\mu_A(x) \mid u = f(x)\}, \sup\{\mu_A(y) \mid v = \varphi(y)\}) \\
&= T(\varphi(\mu_A)(u), \varphi(\mu_A)(v))
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\nu_A)(uv) &= \inf\{\nu_A(xy) \mid u = \varphi(x), v = \varphi(y)\} \\
&\leq \inf\{S(\nu_A(x), \nu_A(y)) \mid u = \varphi(x), v = \varphi(y)\} \\
&= S(\inf\{\nu_A(x) \mid u = f(x)\}, \inf\{\nu_A(y) \mid v = \varphi(y)\}) \\
&= S(\varphi(\nu_A)(u), \varphi(\nu_A)(v)).
\end{aligned}$$

Which mean that

$$\varphi(A)(uv) \supseteq (T(\varphi(\mu_A)(u), \varphi(\mu_A)(v)), S(\varphi(\nu_A)(u), \varphi(\nu_A)(v))).$$

Also

$$\begin{aligned}
\varphi(\mu_A)(u^{-1}) &= \sup\{\mu_A(x^{-1}) \mid u^{-1} = \varphi(x^{-1})\} \\
&= \sup\{\mu_A(x^{-1}) \mid u^{-1} = \varphi^{-1}(x)\} \\
&\geq \sup\{\mu_A(x) \mid u = \varphi(x)\} \\
&= \varphi(\mu_A)(u)
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\nu_A)(u^{-1}) &= \inf\{\nu_A(x^{-1}) \mid u^{-1} = \varphi(x^{-1})\} \\
&= \inf\{\nu_A(x^{-1}) \mid u^{-1} = \varphi^{-1}(x)\} \\
&\leq \inf\{\nu_A(x) \mid u = \varphi(x)\} \\
&= \varphi(\nu_A)(u).
\end{aligned}$$

Thus

$$\varphi(A)(u^{-1}) = (\varphi(\mu_A)(u^{-1}), \varphi(\nu_A)(u^{-1})) \supseteq (\varphi(\mu_A)(u), \varphi(\nu_A)(u)) = \varphi(A)(u).$$

Therefore $\varphi(A) \in IFGN(H)$. $\square$

Proposition 11. *Let H be a group and $B = (\mu_B, \nu_B) \in IFGN(H)$. Suppose that $\varphi : G \to H$ is a homomorphism. Then $\varphi^{-1}(B) \in IFGN(G)$.*

Proof. Let $x, y \in G$ and $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) = (\mu_B(\varphi), \nu_B(\varphi))$. Now

$$\varphi^{-1}(\mu_B)(xy) = \mu_B(\varphi(xy)) = \mu_B(\varphi(x)\varphi(y)) \geq T(\mu_B(\varphi(x)), \mu_B(\varphi(y))) = T(\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\mu_B)(y))$$

and

$$\varphi^{-1}(\nu_B)(xy) = \nu_B(\varphi(xy)) = \nu_B(\varphi(x)\varphi(y)) \leq S(\nu_B(\varphi(x)), \nu_B(\varphi(y))) = S(\varphi^{-1}(\nu_B)(x), \varphi^{-1}(\nu_B)(y)).$$

So

$$\varphi^{-1}(B)(xy) \supseteq (T(\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\mu_B)(y)), S(\varphi^{-1}(\nu_B)(x), \varphi^{-1}(\nu_B)(y))).$$

Also

$$\varphi^{-1}(\mu_B)(x^{-1}) = \mu_B(\varphi(x^{-1})) = \mu_B(\varphi^{-1}(x)) \geq \mu_B(\varphi(x)) = \varphi^{-1}(\mu_B)(x)$$

and

$$\varphi^{-1}(\nu_B)(x^{-1}) = \nu_B(\varphi(x^{-1})) = \nu_B(\varphi^{-1}(x)) \leq \nu_B(\varphi(x)) = \varphi^{-1}(\nu_B)(x).$$

Thus

$$\varphi^{-1}(B)(x^{-1}) = (\varphi^{-1}(\mu_B)(x^{-1}), \varphi^{-1}(\nu_B)(x^{-1})) \supseteq (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = \varphi^{-1}(B)(x).$$

Hence $\varphi^{-1}(B) \in IFGN(G)$. □

Proposition 12. *Let $A = (\mu_A, \nu_A) \in NIFGN(G)$ and H be a group. Suppose that $\varphi : G \to H$ is a homomorphism. Then $\varphi(A) \in NIFGN(H)$.*

Proof. As Proposition 10 we have that $\varphi(A) \in IFGN(H)$. Let $x, y \in H$ such that $\varphi(u) = x$ and $\varphi(w) = y$ with $u, w \in G$. Then

$$\begin{aligned}
\varphi(\mu_A(xyx^{-1})) &= \sup\{\mu_A(w) \mid w \in G, \varphi(w) = xyx^{-1}\} \\
&= \sup\{\mu_A(w) \mid w \in G, \varphi(w) = \varphi(u)\varphi(w)\varphi(u^{-1})\} \\
&= \sup\{\mu_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\} \\
&= \sup\{\mu_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1}) = y\} \\
&= \sup\{\mu_A(w) \mid w \in G, \varphi(w) = y\} \\
&= \varphi(\mu_A(y))
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\nu_A(xyx^{-1})) &= \inf\{\nu_A(w) \mid w \in G, \varphi(w) = xyx^{-1}\} \\
&= \inf\{\nu_A(w) \mid w \in G, \varphi(w) = \varphi(u)\varphi(w)\varphi(u^{-1})\} \\
&= \inf\{\nu_A(w) \mid w \in G, \varphi(w) = \varphi(uwu^{-1})\} \\
&= \inf\{\nu_A(uwu^{-1}) \mid w \in G, \varphi(uwu^{-1}) = y\} \\
&= \inf\{\nu_A(w) \mid w \in G, \varphi(w) = y\} \\
&= \varphi(\nu_A(y)).
\end{aligned}$$

Which yield

$$\varphi(A)(xyx^{-1})) = (\varphi(\mu_A(xyx^{-1})), \varphi(\nu_A(xyx^{-1}))) = (\varphi(\mu_A(y)), \varphi(\nu_A(y))) = \varphi(A)(y)).$$

Thus $\varphi(A) \in NIFGN(H)$. □

Proposition 13. *Let H be a group and $B = (\mu_B, \nu_B) \in NIFGN(H)$. Suppose that $\varphi : G \to H$ is a homomorphism. Then $\varphi^{-1}(B) \in NIFGN(G)$.*

Proof. By Proposition 11 we get that $\varphi^{-1}(B) \in IFGN(G)$. Let $x, y \in G$, then

$$\begin{aligned}\varphi^{-1}(\mu_B)(xyx^{-1}) &= \mu_B(\varphi(xyx^{-1}))\\ &= \mu_B(\varphi(x)\varphi(y)\varphi(x^{-1}))\\ &= \mu_B(\varphi(x)\varphi(y)\varphi^{-1}(x))\\ &= \mu_B(\varphi(y))\\ &= \varphi^{-1}(\mu_B)(y)\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\nu_B)(xyx^{-1}) &= \nu_B(\varphi(xyx^{-1}))\\ &= \nu_B(\varphi(x)\varphi(y)\varphi(x^{-1}))\\ &= \nu_B(\varphi(x)\varphi(y)\varphi^{-1}(x))\\ &= \nu_B(\varphi(y))\\ &= \varphi^{-1}(\nu_B)(y).\end{aligned}$$

Then

$$\varphi^{-1}(B)(xyx^{-1}) = (\varphi^{-1}(\mu_B)(xyx^{-1}), \varphi^{-1}(\nu_B)(xyx^{-1})) = (\varphi^{-1}(\mu_B)(y), \varphi^{-1}(\nu_B)(y)) = \varphi^{-1}(B)(y).$$

Thus $\varphi^{-1}(B) \in NIFGN(G)$. □

Proposition 14. *Let $A = (\mu_A, \nu_A) \in IFGN(G)$ and $B = (\mu_B, \nu_B) \in IFGN(G)$ such that $A \blacktriangleright B$. If $\varphi : G \to H$ is a homomorphism, then $\varphi(A) \blacktriangleright \varphi(B)$.*

Proof. Using Proposition 10 we will have that $\varphi(A) \in IFGN(H)$ and $\varphi(B) \in IFGN(H)$. Let $x, y \in H$ and $u, v \in G$, then

$$\begin{aligned}\varphi(\mu_A)(xyx^{-1}) &= \sup\{\mu_A(z) \mid z \in G, \varphi(z) = xyx^{-1}\}\\ &= \sup\{\mu_A(uvu^{-1}) \mid u, v \in G, \varphi(u) = x, \varphi(v) = y\}\\ &\geq \sup\{T(\mu_A(v), \mu_B(u)) \mid \varphi(u) = x, \varphi(v) = y\}\\ &= T(\sup\{\mu_A(v) \mid y = \varphi(v)\}, \sup\{\mu_B(u) \mid x = \varphi(u)\})\\ &= T(\varphi(\mu_A)(y), \varphi(\mu_B)(x))\end{aligned}$$

and

$$\begin{aligned}\varphi(\nu_A)(xyx^{-1}) &= \inf\{\nu_A(z) \mid z \in G, \varphi(z) = xyx^{-1}\}\\ &= \inf\{\nu_A(uvu^{-1}) \mid u, v \in G, \varphi(u) = x, \varphi(v) = y\}\\ &\leq \inf\{S(\nu_A(v), \nu_B(u)) \mid \varphi(u) = x, \varphi(v) = y\}\\ &= S(\inf\{\nu_A(v) \mid y = \varphi(v)\}, \inf\{\nu_B(u) \mid x = \varphi(u)\})\\ &= S(\varphi(\nu_A)(y), \varphi(\nu_B)(x)).\end{aligned}$$

Then

$$\begin{aligned}\varphi(A)(xyx^{-1}) &= (\varphi(\mu_A)(xyx^{-1}), \varphi(\nu_A)(xyx^{-1}))\\ &\supseteq (T(\varphi(\mu_A)(y), \varphi(\mu_B)(x)), S(\varphi(\nu_A)(y), \varphi(\nu_B)(x))).\end{aligned}$$

Thus $\varphi(A) \blacktriangleright \varphi(B)$. □

Proposition 15. *Let $A = (\mu_A, \nu_A) \in IFGN(H)$ and $B = (\mu_B, \nu_B) \in IFGN(H)$ such that $A \blacktriangleright B$. If $\varphi : G \to H$ is a homomorphism, then $\varphi^{-1}(A) \blacktriangleright \varphi^{-1}(B)$.*

Proof. As Proposition 11 we will have that $\varphi^{-1}(A) \in IFGN(G)$ and $\varphi^{-1}(B) \in IFGN(G)$. Let $x, y \in G$, then

$$\begin{aligned}
\varphi^{-1}(\mu_A)(xyx^{-1}) &= \mu_A(\varphi(xyx^{-1})) \\
&= \mu_A(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
&= \mu_A(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
&\geq T(\mu_A(\varphi(y)), \mu_B(\varphi(x))) \\
&= T(\varphi^{-1}(\mu_A)(y), \varphi^{-1}(\mu_B)(x))
\end{aligned}$$

and

$$\begin{aligned}
\varphi^{-1}(\nu_A)(xyx^{-1}) &= \nu_A(\varphi(xyx^{-1})) \\
&= \nu_A(\varphi(x)\varphi(y)\varphi(x^{-1})) \\
&= \nu_A(\varphi(x)\varphi(y)\varphi^{-1}(x)) \\
&\leq S(\nu_A(\varphi(y)), \nu_B(\varphi(x))) \\
&= S(\varphi^{-1}(\nu_A)(y), \varphi^{-1}(\nu_B)(x)).
\end{aligned}$$

Then

$$\begin{aligned}
\varphi^{-1}(A)(xyx^{-1}) &= (\varphi^{-1}(\mu_A)(xyx^{-1}), \varphi^{-1}(\nu_A)(xyx^{-1})) \\
&\supseteq (T(\varphi^{-1}(\mu_A)(y), \varphi^{-1}(\mu_B)(x)), S(\varphi^{-1}(\nu_A)(y), \varphi^{-1}(\nu_B)(x))).
\end{aligned}$$

Thus $\varphi^{-1}(A) \blacktriangleright \varphi^{-1}(B)$. $\square$

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References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control, 8*(3), 338-353.

[2] Rosenfeld, A. (1971). Fuzzy groups. *Journal of Mathematical Analysis and Applications, 35*(3), 512-517.

[3] Mordeson, J. N., Bhutani, K. R., & Rosenfeld, A. (2005). *Fuzzy group theory* (Vol. 182). New York: Springer.

[4] Mukherjee, N. P., & Bhattacharya, P. (1984). Fuzzy normal subgroups and fuzzy cosets. *Information Sciences, 34*(3), 225-239.

[5] Sessa, S. (1984). On fuzzy subgroups and fuzzy ideals under triangular norms. *Fuzzy Sets and Systems, 13*(1), 95-100.

[6] Osman, M. A. (1987). On some product of fuzzy subgroups. *Fuzzy sets and systems, 24*(1), 79-86.

[7] Anthony, J. M., & Sherwood, H. (1982). A characterization of fuzzy subgroups. *Fuzzy Sets and Systems, 7*(3), 297-305.

[8] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 20*, 87-96.

[9] Zhan, J. I., & Tan, Z. (2004). Intuitionistic M-fuzzy groups. *Soochow Journal of Mathematics, 30*(1), 85-90.

[10] Rasuli, R. (2016). Fuzzy Ideals of Subtraction Semigroups with Respect to A t-norm and A t-conorm. *The Journal of Fuzzy Mathematics Los Angeles, 24*(4), 881-892.

[11] Atanassov, K. T. (1994). New operations defined over the intuitionistic fuzzy sets. *Fuzzy sets and Systems, 61*(2), 137-142.

[12] Atanassov, K. T. (1999). Intuitionistic fuzzy sets. *In Intuitionistic fuzzy sets* (pp. 1-137). Physica, Heidelberg.

[13] Buckley, J. J., & Eslami, E. (2002). *An introduction to fuzzy logic and fuzzy sets* (Vol. 13). Springer Science & Business Media.

[14] Lee, J. (2010). *Introduction to topological manifolds* (Vol. 202). Springer Science & Business Media.

[15] Bojadziev, G., & Bojadziev, M. (1995). *Fuzzy sets, fuzzy logic, applications* (Vol. 5). World scientific.

[16] Rasuli, R. (2017). Characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms by norms. *Journal of New Theory, 18*, 39-52.