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Hermite-Hadamard-Fejér type inequalities for co-ordinated harmonically convex functions via Katugampola fractional integrals

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Abstract: The aim of this paper is to establish the Hermite-Hadamard-Fejér type inequalities for co-ordinated harmonically convex functions via Katugampola fractional integral. We provide Hermite-Hadamard-Fejér inequalities for harmonically convex functions via Katugampola fractional integral in one dimension.

Keywords: Hermite-Hadamard-Fejér inequalities; Riemann-Liouville fractional integral; Katugampola fractional integral; Harmonically convex functions; Co-ordinated harmonically convex functions.

1. Introduction

A function $f:\mathcal{K}\to\mathbb{R}$, where $\mathcal{K}$ is an interval of real numbers, is called convex if the following inequality holds:

$$f(ru_1+(1-r)u_2)\leq rf(u_1)+(1-r)f(u_2), \tag{1}$$

for all $u_1,u_2\in\mathcal{K}$ and $r\in[0,1]$. Function f is called concave if $-f$ is convex.

The Hermite-Hadamard inequality [1] for convex functions $f:\mathcal{K}\to\mathbb{R}$ on an interval of real line is:

$$f\left(\frac{u_1+u_2}{2}\right)\leq\frac{1}{u_2-u_1}\int_{u_1}^{u_2}f(x)dx\leq\frac{f(u_1)+f(u_2)}{2}, \tag{2}$$

where $u_1,u_2\in\mathcal{K}$ with $u_1<u_2$. Then Fejér [2] introduced the weighted generalization of (2) as follows

$$f\left(\frac{u_1+u_2}{2}\right)\int_{u_1}^{u_2}g(x)dx\leq\frac{1}{u_2-u_1}\int_{u_1}^{u_2}f(x)g(x)dx\leq\frac{f(u_1)+f(u_2)}{2}\int_{u_1}^{u_2}g(x)dx, \tag{3}$$

where $g:[u_1,u_2]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $(u_1+u_2)/2$. For more results and details see [3–14].

Definition 1 ([15])**.** Let $\mathcal{K}\subset\mathbb{R}\setminus\{0\}$ be a real interval. A function $f:\mathcal{K}\to\mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{u_1u_2}{ru_1+(1-r)u_2}\right)\leq rf(u_2)+(1-r)f(u_1), \tag{4}$$

for all $u_1,u_2\in\mathcal{K}$ and $r\in[0,1]$. If the inequality in (4) is reversed, then f is said to be harmonically concave.

Dragomir [16] gave the Hadamard's inequality for convex functions on the co-ordinate which is defined as:

Definition 2 ([16])**.** A function $f:\Delta=[u_1,u_2]\times[v_1,v_2]\subseteq\mathbb{R}^2\to\mathbb{R}$ is called convex on the co-ordinate with $u_1<u_2$ and $v_1<v_2$ if the partial functions

$$f_y:[u_1,u_2]\to\mathbb{R},\ f_y(a)=f(a,y)\text{ and }f_x:[v_1,v_2]\to\mathbb{R},\ f_x(c)=f(x,c)$$

are convex for all $x\in[u_1,u_2]$ and $y\in[v_1,v_2]$.

Definition 3 ([17]). A function $f:\Delta=[u_1,u_2]\times[v_1,v_2]\subseteq\mathbb{R}^2\to\mathbb{R}$ is called co-ordinate convex on Δ with $u_1<u_2$ and $v_1<v_2$, if

$$f(rx+(1-r)z,\tau y+(1-\tau)w)\leq r\tau f(x,y)+r(1-\tau)f(x,w)+(1-r)\tau f(z,y)+(1-r)(1-\tau)f(z,w),$$

for all $r,\tau\in[0,1]$ and $(x,y),(z,w)\in\Delta$. For more results and details see [16–19].

Definition 4 ([20]). A function $f:\Delta=[u_1,u_2]\times[v_1,v_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ is called co-ordinated harmonically convex on Δ with $u_1<u_2$ and $v_1<v_2$, if

$$f\left(\frac{xz}{rx+(1-r)z},\frac{yw}{\tau y+(1-\tau)w}\right)\leq r\tau f(x,y)+r(1-\tau)f(x,w)+(1-r)\tau f(z,y)+(1-r)(1-\tau)f(z,w),$$

for all $r,\tau\in[0,1]$ and $(x,y),(z,w)\in\Delta$.

Clearly, a function $f:\Delta=[u_1,u_2]\times[v_1,v_2]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ is called harmonically convex on the co-ordinate with $u_1<u_2$ and $v_1<v_2$ if the partial functions

$$f_y:[u_1,u_2]\to\mathbb{R},\ f_y(a)=f(a,y)\text{ and }f_x:[v_1,v_2]\to\mathbb{R},\ f_x(b)=f(x,b)$$

are harmonically convex for all $x\in[u_1,u_2]$ and $y\in[v_1,v_2]$, see [21] for more details.

Definition 5 ([22]). Let $[u_1,u_2]\subset\mathbb{R}$ be a finite interval. The left- and right-side Katugampola fractional integrals of order $\alpha(>0)$ of $f\in X_c^p(u_1,u_2)$ are defined by,

$${}^{\rho}I^{\alpha}_{u_1+}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{u_1}^{x}(x^{\rho}-t^{\rho})^{\alpha-1}t^{\rho-1}f(t)dt,$$

and

$${}^{\rho}I^{\alpha}_{u_2-}f(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{u_2}(t^{\rho}-x^{\rho})^{\alpha-1}t^{\rho-1}f(t)dt,$$

with $u_1<x<u_2$ and $\rho>0$, where $X_c^p(u_1,u_2)$ $(c\in\mathbb{R},1\leq p\leq\infty)$ is the space of those complex valued Lebesgue measurable functions f on $[u_1,u_2]$ for which $\|f\|_{X_c^p}<\infty$, where the norm is defined by

$$\|f\|_{X_c^p}=\left(\int_{u_1}^{u_2}|t^cf(t)|^p\frac{dt}{t}\right)^{1/p}<\infty,$$

for $1\leq p<\infty$, $c\in\mathbb{R}$ and for the case $p=\infty$,

$$\|f\|_{X_c^\infty}=ess\ sup_{u_1\leq t\leq u_2}[t^c|f(t)|].$$

Definition 6 ([23]). Let $f\in L_1([u_1,u_2]\times[v_1,v_2])$. The Katugampola fractional integrals ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_1+,v_1+}$, ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_1+,v_2-}$, ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_2-,v_1+}$ and ${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_2-,v_2-}$ of order $\alpha,\beta>0$ with $a,c\geq0$ are defined by

$${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_1+,v_1+}f(x,y)=\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_{u_1}^{x}\int_{v_1}^{y}(x^{\rho_1}-t^{\rho_1})^{\alpha-1}(y^{\rho_2}-s^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}f(t,s)dsdt,$$

with $x>u_1,\ y>v_1$,

$${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_1+,v_2-}f(x,y)=\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_{u_1}^{x}\int_{y}^{v_2}(x^{\rho_1}-t^{\rho_1})^{\alpha-1}(s^{\rho_2}-y^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}f(t,s)dsdt,$$

with $x>u_1,\ y<v_2$,

$${}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_2-,v_1+}f(x,y)=\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{u_2}\int_{v_1}^{y}(t^{\rho_1}-x^{\rho_1})^{\alpha-1}(y^{\rho_2}-s^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}f(t,s)dsdt,$$

with $x < u_2,\ y > v_1$, and

$$ {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{u_2-,v_2-}f(x,y) = \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_x^{u_2}\int_y^{v_2}(t^{\rho_1}-x^{\rho_1})^{\alpha-1}(s^{\rho_2}-y^{\rho_2})^{\beta-1}t^{\rho_1-1}s^{\rho_2-1}f(t,s)dsdt, $$

with $x < u_2,\ y < v_2$, respectively, where the Gamma function Γ is defined as $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

In the next section, we give result for harmonically convex functions in one dimension.

2. Hermite-Hadamard-Fejér type inequalities

In this section, we give Hermite-Hadamard-Fejér type inequalities for harmonically convex functions via Katugampola fractional integral in one dimension which will play a key role for the results in the next section. Latif *et al.*, [24] defined following useful definition:

Definition 7 ([24]). A function $h : [u_1, u_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $2u_1u_2/(u_1+u_2)$ if

$$ h(x) = h\left(\frac{1}{\frac{1}{u_1}+\frac{1}{u_2}-\frac{1}{x}}\right) $$

holds for all $x \in [u_1, u_2]$.

Lemma 1. *Let $\rho > 0$. If $h : [u_1^\rho, u_2^\rho] \subseteq (0,\infty) \to \mathbb{R}$ is integrable and harmonically symmetric with respect to $2u_1^\rho u_2^\rho/(u_1^\rho+u_2^\rho)$, then*

$$ {}^\rho I^\alpha_{1/u_2+}(h\circ g)(1/u_1^\rho) = {}^\rho I^\alpha_{1/u_1-}(h\circ g)(1/u_2^\rho) = \frac{1}{2}\left[{}^\rho I^\alpha_{1/u_2+}(h\circ g)(1/u_1^\rho) + {}^\rho I^\alpha_{1/u_1-}(h\circ g)(1/u_2^\rho)\right], \quad (5) $$

with $\alpha > 0$ and $g(x^\rho) = 1/x^\rho$.

Proof. Since h is harmonically symmetric with respect to $2u_1^\rho u_2^\rho/(u_1^\rho+u_2^\rho)$, then by definition we have $h(\frac{1}{x^\rho}) = h\left(\frac{1}{\frac{1}{u_1^\rho}+\frac{1}{u_2^\rho}-x^\rho}\right)$, for all $x^\rho \in \left[\frac{1}{u_2^\rho}, \frac{1}{u_1^\rho}\right]$. In the following integral, by setting $t^\rho = \frac{1}{u_1^\rho}+\frac{1}{u_2^\rho}-x^\rho$, we get

$$ \begin{aligned} {}^\rho I^\alpha_{1/u_2+}(h\circ g)(1/u_1^\rho) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\frac{1}{u_2}}^{\frac{1}{u_1}}\left(\frac{1}{u_1^\rho}-t^\rho\right)^{\alpha-1}t^{\rho-1}h\left(\frac{1}{t^\rho}\right)dt \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\frac{1}{u_2}}^{\frac{1}{u_1}}\left(x^\rho-\frac{1}{u_2^\rho}\right)^{\alpha-1}x^{\rho-1}h\left(\frac{1}{\frac{1}{u_1^\rho}+\frac{1}{u_2^\rho}-x^\rho}\right)dx \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\frac{1}{u_2}}^{\frac{1}{u_1}}\left(x^\rho-\frac{1}{u_2^\rho}\right)^{\alpha-1}x^{\rho-1}h\left(\frac{1}{x^\rho}\right)dx \\ &= {}^\rho I^\alpha_{1/u_1-}(h\circ g)(1/u_2^\rho). \end{aligned} $$

This completes the proof. □

Remark 1. In Lemma 1, if we take $\rho \longmapsto 0$, we get Lemma 2 in [25].

Theorem 8. *Let $\rho > 0$. Let $f : [u_1^\rho, u_2^\rho] \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically convex with $u_1 < u_2$ and $f \in L_1[u_1, u_2]$. If $h : [u_1^\rho, u_2^\rho] \subseteq (0,\infty) \to \mathbb{R}$ is nonnegative and harmonically symmetric with respect to $2u_1^\rho u_2^\rho/(u_1^\rho+u_2^\rho)$, then the following inequalities hold:*

$$ f\left(\frac{2u_1^\rho u_2^\rho}{u_1^\rho+u_2^\rho}\right)\left[{}^\rho I^\alpha_{1/u_1-}(h\circ g)(1/u_2^\rho) + {}^\rho I^\alpha_{1/u_2+}(h\circ g)(1/u_1^\rho)\right] \leq \left[{}^\rho I^\alpha_{1/u_1-}(fh\circ g)(1/u_2^\rho) + {}^\rho I^\alpha_{1/u_2+}(fh\circ g)(1/u_1^\rho)\right] $$

$$\leq \frac{f(u_1^\rho)+f(u_2^\rho)}{2}\left[{}^\rho I^\alpha_{1/u_1-}(h\circ g)(1/u_2^\rho) +{}^\rho I^\alpha_{1/u_2+}(h\circ g)(1/u_1^\rho)\right], \tag{6}$$

with $\alpha>0$ *and* $g(x^\rho)=1/x^\rho$.

Proof. Since f is harmonically convex on $[u_1^\rho,u_2^\rho]$, we have for all $r\in[0,1]$

$$\begin{aligned} f\left(\frac{2u_1^\rho u_2^\rho}{u_1^\rho+u_2^\rho}\right) &= f\left(\frac{2u_1^\rho u_2^\rho}{\left(r^\rho u_1^\rho+(1-r^\rho)u_2^\rho\right)+\left(r^\rho u_2^\rho+(1-r^\rho)u_1^\rho\right)}\right)\\ &\leq \frac{f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_1^\rho+(1-r^\rho)u_2^\rho}\right)+f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)}{2}. \end{aligned} \tag{7}$$

Multiplying (7) by $r^{\alpha\rho-1}h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)$ on both sides and integrate with respect to $[0,1]$, we get

$$\begin{aligned} &2f\left(\frac{2u_1^\rho u_2^\rho}{u_1^\rho+u_2^\rho}\right)\int_0^1 r^{\alpha\rho-1}h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)dr\\ &\leq \int_0^1 r^{\alpha\rho-1}f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_1^\rho+(1-r^\rho)u_2^\rho}\right)h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)dr\\ &\quad+\int_0^1 r^{\alpha\rho-1}f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)dr. \end{aligned}$$

Since h is harmonically symmetric with respect to $2u_1^\rho u_2^\rho/(u_1^\rho+u_2^\rho)$. By setting $x^\rho=\frac{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}{u_1^\rho u_2^\rho}$, we get

$$\begin{aligned} &2\left(\frac{u_1^\rho u_2^\rho}{u_2^\rho-u_1^\rho}\right)^\alpha f\left(\frac{2u_1^\rho u_2^\rho}{u_1^\rho+u_2^\rho}\right)\int_{1/u_2}^{1/u_1}\left(x^p-\frac{1}{u_2^\rho}\right)^{\alpha-1}h\left(\frac{1}{x^\rho}\right)dx\\ &\leq \left(\frac{u_1^\rho u_2^\rho}{u_2^\rho-u_1^\rho}\right)^\alpha\left[\int_{1/u_2}^{1/u_1}\left(x^p-\frac{1}{u_2^\rho}\right)^{\alpha-1}f\left(\frac{1}{\frac{1}{u_1^\rho}+\frac{1}{u_2^\rho}-x^\rho}\right)h\left(\frac{1}{x^\rho}\right)dx\right.\\ &\quad\left.+\int_{1/u_2}^{1/u_1}\left(x^p-\frac{1}{u_2^\rho}\right)^{\alpha-1}f\left(\frac{1}{x^\rho}\right)h\left(\frac{1}{x^\rho}\right)dx\right]\\ &= \left(\frac{u_1^\rho u_2^\rho}{u_2^\rho-u_1^\rho}\right)^\alpha\left[\int_{1/u_2}^{1/u_1}\left(\frac{1}{u_1^\rho}-x^p\right)^{\alpha-1}f\left(\frac{1}{x^\rho}\right)h\left(\frac{1}{\frac{1}{u_1^\rho}+\frac{1}{u_2^\rho}-x^\rho}\right)dx\right.\\ &\quad\left.+\int_{1/u_2}^{1/u_1}\left(x^p-\frac{1}{u_2^\rho}\right)^{\alpha-1}f\left(\frac{1}{x^\rho}\right)h\left(\frac{1}{x^\rho}\right)dx\right]. \end{aligned}$$

Then by Lemma 1, we have

$$\begin{aligned} &\left(\frac{u_1^\rho u_2^\rho}{u_2^\rho-u_1^\rho}\right)^\alpha\rho^{\alpha-1}\Gamma(\alpha)f\left(\frac{2u_1^\rho u_2^\rho}{u_1^\rho+u_2^\rho}\right)\left[{}^\rho I^\alpha_{1/u_1-}(h\circ g)(1/u_2^\rho)+{}^\rho I^\alpha_{1/u_2+}(h\circ g)(1/u_1^\rho)\right]\\ &\leq \left(\frac{u_1^\rho u_2^\rho}{u_2^\rho-u_1^\rho}\right)^\alpha\rho^{\alpha-1}\Gamma(\alpha)\left[{}^\rho I^\alpha_{1/u_1-}(fh\circ g)(1/u_2^\rho)+{}^\rho I^\alpha_{1/u_2+}(fh\circ g)(1/u_1^\rho)\right]. \end{aligned} \tag{8}$$

This completes the first inequality. For second inequality, we first note that if f is harmonically convex function, then we have

$$f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_1^\rho+(1-r^\rho)u_2^\rho}\right)+f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)\leq f(u_1^\rho)+f(u_2^\rho). \tag{9}$$

Multiplying (8) by $r^{\alpha\rho-1}h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)$ on both sides and integrate with respect to $r\in[0,1]$, we get

$$\begin{aligned}
&\int_0^1 r^{\alpha\rho-1}f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_1^\rho+(1-r^\rho)u_2^\rho}\right)h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)dr\\
&+\int_0^1 r^{\alpha\rho-1}f\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)dr\\
&\leq (f(u_1^\rho)+f(u_2^\rho))\int_0^1 r^{\alpha\rho-1}h\left(\frac{u_1^\rho u_2^\rho}{r^\rho u_2^\rho+(1-r^\rho)u_1^\rho}\right)dr,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left(\frac{u_1^\rho u_2^\rho}{u_2^\rho-u_1^\rho}\right)^\alpha \rho^{\alpha-1}\Gamma(\alpha)\left[{}^\rho I^\alpha_{1/u_1-}(fh\circ g)(1/u_2^\rho)+{}^\rho I^\alpha_{1/u_2+}(fh\circ g)(1/u_1^\rho)\right]\\
&\leq \left(\frac{u_1^\rho u_2^\rho}{u_2^\rho-u_1^\rho}\right)^\alpha \rho^{\alpha-1}\Gamma(\alpha)\frac{f(u_1^\rho)+f(u_2^\rho)}{2}\left[{}^\rho I^\alpha_{1/u_1-}(h\circ g)(1/u_2^\rho)+{}^\rho I^\alpha_{1/u_2+}(h\circ g)(1/u_1^\rho)\right].
\end{aligned}$$

This completes the proof. □

Remark 2. 1) In Theorem 8, if we take $\rho\to 1$, we get Theorem 5 in [25].
2) In Theorem 8, if we take $\rho\to 1$ and $\alpha=1$, we get Theorem 8 in [26].

3. Hermite-Hadamard-Fejér type inequalities on co-ordinates

In this section, we established some new results by using Katugampola fractional integrals on co-ordinates. First we give the following result:

Theorem 9. *Let $\alpha,\beta>0$ and $\rho_1,\rho_2>0$. Let $f:\Delta=[u_1^{\rho_1},u_2^{\rho_1}]\times[v_1^{\rho_2},v_2^{\rho_2}]\subseteq(0,\infty)\times(0,\infty)\to\mathbb{R}$ be a co-ordinated harmonically convex on Δ, with $0<u_1<u_2$, $0<v_1<v_2$. If $h:\Delta\to\mathbb{R}$ is nonnegative and harmonically symmetric with respect to $\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}$ on Δ. Then*

$$\begin{aligned}
&f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\\
&\quad+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\Bigg]\\
&\leq\frac{1}{4}\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}(fh\circ g)\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}(fh\circ g)\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\\
&\quad+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}(fh\circ g)\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}(fh\circ g)\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\Bigg]\\
&\leq\frac{f(u_1^{\rho_1},v_1^{\rho_2})+f(u_1^{\rho_1},v_2^{\rho_2})+f(u_2^{\rho_1},v_1^{\rho_2})+f(u_2^{\rho_1},v_2^{\rho_2})}{4}\\
&\quad\times\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\\
&\quad+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\Bigg],
\end{aligned}\tag{10}$$

holds, where $g(x^{\rho_1},y^{\rho_2})=\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)$.

Proof. Since f is co-ordinated harmonically convex on Δ, we have

$$\begin{aligned} f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right) \leq & \frac{1}{4}\left[f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_1^{\rho_1}+(1-r^{\rho_1})u_2^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_1^{\rho_2}+(1-\tau^{\rho_2})v_2^{\rho_2}}\right)\right. \\ & + f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_1^{\rho_1}+(1-r^{\rho_1})u_2^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right) \\ & + f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_1^{\rho_2}+(1-\tau^{\rho_2})v_2^{\rho_2}}\right) \\ & \left.+ f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)\right]. \end{aligned} \tag{11}$$

Multiplying (11) by $r^{\rho_1\alpha-1}\tau^{\rho_2\beta-1}h\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)$ on both sides and then integrating with respect to (r,τ) over $[0,1]\times[0,1]$, we get

$$\begin{aligned} & f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\int_0^1\int_0^1 r^{\rho_1\alpha-1}\tau^{\rho_2\beta-1}h\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)drd\tau \\ \leq & \frac{1}{4}\left[\int_0^1\int_0^1 f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_1^{\rho_1}+(1-r^{\rho_1})u_2^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_1^{\rho_2}+(1-\tau^{\rho_2})v_2^{\rho_2}}\right)\right. \\ & \times h\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)r^{\rho_1\alpha-1}\tau^{\rho_2\beta-1}drd\tau \\ & + \int_0^1\int_0^1 f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_1^{\rho_1}+(1-r^{\rho_1})u_2^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right) \\ & \times h\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)r^{\rho_1\alpha-1}\tau^{\rho_2\beta-1}drd\tau \\ & + \int_0^1\int_0^1 f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_1^{\rho_2}+(1-\tau^{\rho_2})v_2^{\rho_2}}\right) \\ & \times h\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)r^{\rho_1\alpha-1}\tau^{\rho_2\beta-1}drd\tau \\ & + \int_0^1\int_0^1 f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right) \\ & \left.\times h\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)r^{\rho_1\alpha-1}\tau^{\rho_2\beta-1}drd\tau\right]. \end{aligned}$$

By change of variables $x^{\rho_1}=\frac{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}}{u_1^{\rho_1}u_2^{\rho_1}}$ and $y^{\rho_2}=\frac{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}{v_1^{\rho_2}v_2^{\rho_2}}$ and using the symmetric property of h, we find

$$\begin{aligned} & \left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{u_2^{\rho_1}-u_1^{\rho_1}}\right)^{\alpha}\left(\frac{v_1^{\rho_2}v_2^{\rho_2}}{v_2^{\rho_2}-v_1^{\rho_2}}\right)^{\beta} f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right) \\ & \times\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dxdy \\ \leq & \frac{1}{4}\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{u_2^{\rho_1}-u_1^{\rho_1}}\right)^{\alpha}\left(\frac{v_1^{\rho_2}v_2^{\rho_2}}{v_2^{\rho_2}-v_1^{\rho_2}}\right)^{\beta}\left[\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}\right. \end{aligned}$$

$$
\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{\frac{1}{u_1^{\rho_1}}+\frac{1}{u_2^{\rho_1}}-x^{\rho_1}},\frac{1}{\frac{1}{v_1^{\rho_2}}+\frac{2}{v_2^{\rho_2}}-y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dxdy
$$

$$
+\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{\frac{1}{u_1^{\rho_1}}+\frac{1}{u_2^{\rho_1}}-x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dxdy
$$

$$
+\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{\frac{1}{v_1^{\rho_2}}+\frac{2}{v_1^{\rho_2}}-y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dxdy
$$

$$
\left.+\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dxdy\right]
$$

$$
=\frac{1}{4}\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{u_2^{\rho_1}-u_1^{\rho_1}}\right)^{\alpha}\left(\frac{v_1^{\rho_2}v_2^{\rho_2}}{v_2^{\rho_2}-v_1^{\rho_2}}\right)^{\beta}\left[\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\right.
$$

$$
\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{\frac{1}{u_1^{\rho_1}}+\frac{1}{u_2^{\rho_1}}-x^{\rho_1}},\frac{1}{\frac{1}{v_1^{\rho_2}}+\frac{2}{v_2^{\rho_2}}-y^{\rho_2}}\right)dxdy
$$

$$
+\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{\frac{1}{u_1^{\rho_1}}+\frac{1}{u_2^{\rho_1}}-x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dxdy
$$

$$
+\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{\frac{1}{v_1^{\rho_2}}+\frac{2}{v_2^{\rho_2}}-y^{\rho_2}}\right)dxdy
$$

$$
\left.+\int_{1/v_2}^{1/v_1}\int_{1/u_2}^{1/u_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dxdy\right].
$$

Thus, we get

$$
\begin{aligned}
&\frac{\Gamma(\alpha)\Gamma(\beta)}{\rho_1^{1-\alpha}\rho_2^{1-\beta}}\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{u_2^{\rho_1}-u_1^{\rho_1}}\right)^{\alpha}\left(\frac{v_1^{\rho_2}v_2^{\rho_2}}{v_2^{\rho_2}-v_1^{\rho_2}}\right)^{\beta}f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\\
&\qquad\times\left[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right.\\
&\qquad\left.+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&\le\frac{\Gamma(\alpha)\Gamma(\beta)}{4\rho_1^{1-\alpha}\rho_2^{1-\beta}}\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{u_2^{\rho_1}-u_1^{\rho_1}}\right)^{\alpha}\left(\frac{v_1^{\rho_2}v_2^{\rho_2}}{v_2^{\rho_2}-v_1^{\rho_2}}\right)^{\beta}\\
&\qquad\times\left[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}(fh\circ g)\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}(fh\circ g)\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right.\\
&\qquad\left.+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}(fh\circ g)\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}(fh\circ g)\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right].
\end{aligned}
$$

This completes the first inequality of (10). For the second inequality of (10) we use the co-ordinated harmonically convexity of f as:

$$
f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_1^{\rho_1}+(1-r^{\rho_1})u_2^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_1^{\rho_2}+(1-\tau^{\rho_2})v_2^{\rho_2}}\right)+f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_1^{\rho_1}+(1-r^{\rho_1})u_2^{\rho_1}},\frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)
$$

$$+ f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}}, \frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_1^{\rho_2}+(1-\tau^{\rho_2})v_2^{\rho_2}}\right) + f\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}}, \frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)$$
$$\le f(u_1^{\rho_1}, v_1^{\rho_2}) + f(u_2^{\rho_1}, v_1^{\rho_2}) + f(u_1^{\rho_1}, v_2^{\rho_2}) + f(u_2^{\rho_1}, v_2^{\rho_2}). \tag{12}$$

Thus multiplying (12) by $r^{\rho_1\alpha-1}\tau^{\rho_2\beta-1}h\left(\frac{u_1^{\rho_1}u_2^{\rho_1}}{r^{\rho_1}u_2^{\rho_1}+(1-r^{\rho_1})u_1^{\rho_1}}, \frac{v_1^{\rho_2}v_2^{\rho_2}}{\tau^{\rho_2}v_2^{\rho_2}+(1-\tau^{\rho_2})v_1^{\rho_2}}\right)$ and integrating with respect to (r,τ) over $[0,1]\times[0,1]$, we get the second inequality of (10). Hence the proof is completed. □

Theorem 10. *Let $\alpha,\beta > 0$ and $\rho_1,\rho_2 > 0$. Let $f : \Delta = [u_1^{\rho_1}, u_2^{\rho_1}] \times [v_1^{\rho_2}, v_2^{\rho_2}] \subseteq (0,\infty)\times(0,\infty) \to \mathbb{R}$ be a co-ordinated harmonically convex on Δ, with $0 < u_1 < u_2$, $0 < v_1 < v_2$ and $f \in L_1[\Delta]$. If $h : \Delta \to \mathbb{R}$ is nonnegative and harmonically symmetric with respect to $\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}}, \frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}$ on Δ. Then the following inequalities hold:*

$$\begin{aligned}
f&\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}}, \frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}h\circ g\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}h\circ g\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}h\circ g\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}h\circ g\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\Bigg] \\
&\le {}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}}, \frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}}, \frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}}, \frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}}, \frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\right] \\
&\le 2\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}fh\circ g\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}fh\circ g\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right) \\
&\quad + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}fh\circ g\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right) + {}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}fh\circ g\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\Bigg] \\
&\le {}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}}, v_1^{\rho_2}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}}, v_2^{\rho_2}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_2^{\rho_1}}, \frac{1}{v_2^{\rho_2}}\right)\right] \\
&\quad + {}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}}, v_1^{\rho_2}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_1^{\rho_1}}, \frac{1}{v_1^{\rho_2}}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ {}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}},v_2^{\rho_2}\right)\,{}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(u_1^{\rho_1},\frac{1}{v_2^{\rho_2}}\right)\,{}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(u_2^{\rho_1},\frac{1}{v_2^{\rho_2}}\right)\,{}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(u_1^{\rho_1},\frac{1}{v_1^{\rho_2}}\right)\,{}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&+ {}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(u_2^{\rho_1},\frac{1}{v_1^{\rho_2}}\right)\,{}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
\le&\frac{f(u_1^{\rho_1},v_1^{\rho_2})+f(u_1^{\rho_1},v_2^{\rho_2})+f(u_2^{\rho_1},v_1^{\rho_2})+f(u_2^{\rho_1},v_2^{\rho_2})}{4}\\
&\times\left[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right.\\
&\left.+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right], \qquad (13)
\end{aligned}
$$

where $g(x^{\rho_1},y^{\rho_2})=\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)$, $g_1(x^{\rho_1},y^{\rho_2})=\left(\frac{1}{x^{\rho_1}},y^{\rho_2}\right)$ *and* $g_2(x^{\rho_1},y^{\rho_2})=\left(x^{\rho_1},\frac{1}{y^{\rho_2}}\right)$, *respectively.*

Proof. Since f is co-ordinated harmonically convex on Δ, then the function $f_{1/x^{\rho_1}}:[v_1^{\rho_2},v_2^{\rho_2}]\to\mathbb{R}$, defined by $f_{1/x^{\rho_1}}(y^{\rho_2})=f(\frac{1}{x^{\rho_1}},y^{\rho_2})$ is harmonically convex on $[v_1^{\rho_2},v_2^{\rho_2}]$ for all $x^{\rho_1}\in\left[\frac{1}{u_2^{\rho_1}},\frac{1}{u_1^{\rho_1}}\right]$. Then from (6), we have

$$
\begin{aligned}
&\frac{\rho_2^{1-\beta}}{\Gamma(\beta)}f\left(\frac{1}{x^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\left[\int_{1/v_2}^{1/v_1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dy\right.\\
&\qquad\left.+\int_{1/v_2}^{1/v_1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dy\right]\\
&\le\frac{\rho_2^{1-\beta}}{\Gamma(\beta)}\left[\int_{1/v_2}^{1/v_1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}y^{\rho_2-1}fh\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dy\right.\\
&\qquad\left.+\int_{1/v_2}^{1/v_1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}y^{\rho_2-1}fh\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dy\right]\\
&\le\frac{f\left(\frac{1}{x^{\rho_1}},v_1^{\rho_2}\right)+f\left(\frac{1}{x^{\rho_1}},v_2^{\rho_2}\right)}{2}\left[\int_{1/v_2}^{1/v_1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dy\right.\\
&\qquad\left.+\int_{1/v_2}^{1/v_1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dy\right]. \qquad (14)
\end{aligned}
$$

Multiplying both sides of (14) by $\frac{x^{\rho_1-1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}}{\rho_1^{\alpha-1}\Gamma(\alpha)}$ and $\frac{x^{\rho_1-1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}}{\rho_1^{\alpha-1}\Gamma(\alpha)}$, and integrating with respect to x over $\left[\frac{1}{u_2},\frac{1}{u_1}\right]$, respectively, we get

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\right.\\
&\quad\times h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\\
&\quad\left.\times f\left(\frac{1}{x^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(\frac{1}{x^{\rho_1}},v_1^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},v_1^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\\
&\quad\times\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},v_2^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\\
&\quad\left.+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},v_2^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right],\quad(15)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\right.\\
&\quad\times h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\\
&\quad\left.\times f\left(\frac{1}{x^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
& \times f\left(\frac{1}{x^{\rho_1}},v_1^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
& \times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},v_1^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\\
& \times\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},v_2^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\\
& +\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\\
& \left.\times f\left(\frac{1}{x^{\rho_1}},v_2^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]. \qquad (16)
\end{aligned}
$$

Using similar arguments for the mapping $f_{\frac{1}{y^{\rho_2}}}:[u_1^{\rho_1},u_2^{\rho_1}]\to\mathbb{R}$, $f_{\frac{1}{y^{\rho_2}}}(x^{\rho_1})=f(x^{\rho_1},\frac{1}{y^{\rho_2}})$, we have

$$
\begin{aligned}
& \frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{y^{\rho_2}}\right)\right.\\
& \quad\times h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\\
& \quad\left.\times f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
& \leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
& \quad\times f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}\\
& \quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
& \leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
& \quad\times f\left(u_1^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}\\
& \quad\times x^{\rho_1-1}y^{\rho_2-1}f\left(u_1^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\\
& \quad\times\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(u_2^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\\
& \quad+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\\
& \quad\left.\times f\left(u_2^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right], \qquad (17)
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{y^{\rho_2}}\right)\right.\\
&\quad\times h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\\
&\quad\left.\times f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(u_1^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\times x^{\rho_1-1}y^{\rho_2-1}f\left(u_1^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\\
&\quad\times\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(b^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\\
&\quad\left.+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}f\left(u_2^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right].
\end{aligned}\tag{18}$$

By adding the inequalities (15)∼(18), we get

$$\begin{aligned}
&{}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ {}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
\leq& 2\left[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}fh\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}fh\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right.\\
&\left.+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}fh\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}fh\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
\leq& {}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}},v_1^{\rho_2}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]+{}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}},v_2^{\rho_2}\right)\right.\\
&\left.\times{}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]+{}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}},v_1^{\rho_2}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&+{}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}},v_2^{\rho_2}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]+{}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(u_1^{\rho_1},\frac{1}{v_2^{\rho_2}}\right)\right.\\
&\left.\times{}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]+{}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(u_2^{\rho_1},\frac{1}{v_2^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&+{}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(u_1^{\rho_1},\frac{1}{v_1^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]+{}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(u_2^{\rho_1},\frac{1}{v_1^{\rho_2}}\right)\right.\\
&\left.\times{}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right].
\end{aligned}
$$

This completes the second and third inequality of (13). Now, using the first inequality of (6), we find

$$
\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\right.\\
&\quad\times x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(\frac{1}{x^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]. \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\right.\\
&\quad\times x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.
\end{aligned}
$$

$$\times f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}$$
$$\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\Bigg]. \quad (20)$$

Adding (19) and (20) and using the fact that h is symmetric, we get

$$\begin{aligned}
&f\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right)\Bigg[{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_1-}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_1-,1/v_2+}h\circ g\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\\
&\qquad+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_1-}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)+{}^{\rho_1,\rho_2}I^{\alpha,\beta}_{1/u_2+,1/v_2+}h\circ g\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\Bigg]\\
&\le {}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_1}I^{\alpha}_{1/u_1-}\left[f\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_1-}h\circ g_2\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_1}I^{\alpha}_{1/u_2+}\left[f\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{2v_1^{\rho_2}v_2^{\rho_2}}{v_1^{\rho_2}+v_2^{\rho_2}}\right){}^{\rho_2}I^{\beta}_{1/v_2+}h\circ g_2\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_2}I^{\beta}_{1/v_1-}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_2^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_1-}h\circ g_1\left(\frac{1}{u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right]\\
&\quad+{}^{\rho_2}I^{\beta}_{1/v_2+}\left[f\circ g_2\left(\frac{2u_1^{\rho_1}u_2^{\rho_1}}{u_1^{\rho_1}+u_2^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right){}^{\rho_1}I^{\alpha}_{1/u_2+}h\circ g_1\left(\frac{1}{u_1^{\rho_1}},\frac{1}{v_1^{\rho_2}}\right)\right].
\end{aligned}$$

This completes the first inequality of (13). Now, to achieve the last inequality of (13), applying the second inequality of (6) as:

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\\
&\quad\times f\left(\frac{1}{x^{\rho_1}},v_2^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}\\
&\quad\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},v_2^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\Bigg]\\
&\le\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\frac{f(u_1^{\rho_1},v_2^{\rho_2})+f(u_2^{\rho_1},v_2^{\rho_2})}{2}\\
&\quad\times\Bigg[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\\
&\quad+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\Bigg], \quad (21)
\end{aligned}$$

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(\frac{1}{x^{\rho_1}},v_1^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(\frac{1}{x^{\rho_1}},v_1^{\rho_2}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\quad\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\frac{f(u_1^{\rho_1},v_1^{\rho_2})+f(u_2^{\rho_1},v_1^{\rho_2})}{2}\\
&\quad\times\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right.\\
&\quad\left.+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right], \qquad (22)
\end{aligned}$$

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(u_2^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(u_2^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\quad\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\frac{f(u_2^{\rho_1},v_1^{\rho_2})+f(u_2^{\rho_1},v_2^{\rho_2})}{2}\\
&\quad\times\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right.\\
&\quad\left.+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(x^{\rho_1}-\frac{1}{u_2^{\rho_1}}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right], \qquad (23)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{2\Gamma(\alpha)\Gamma(\beta)}\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}\right.\\
&\quad\times f\left(u_1^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}\\
&\quad\left.\times x^{\rho_1-1}y^{\rho_2-1}f\left(u_1^{\rho_1},\frac{1}{y^{\rho_2}}\right)h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]\\
&\quad\leq\frac{\rho_1^{1-\alpha}\rho_2^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\frac{f(u_1^{\rho_1},v_1^{\rho_2})+f(u_1^{\rho_1},v_2^{\rho_2})}{2}\\
&\quad\times\left[\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(y^{\rho_2}-\frac{1}{v_2^{\rho_2}}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right.\\
&\quad\left.+\int_{1/u_2}^{1/u_1}\int_{1/v_2}^{1/v_1}\left(\frac{1}{u_1^{\rho_1}}-x^{\rho_1}\right)^{\alpha-1}\left(\frac{1}{v_1^{\rho_2}}-y^{\rho_2}\right)^{\beta-1}x^{\rho_1-1}y^{\rho_2-1}h\left(\frac{1}{x^{\rho_1}},\frac{1}{y^{\rho_2}}\right)dydx\right]. \qquad (24)
\end{aligned}$$

By adding the inequalities (21)∼(24), we get the last inequality of (13). □

Remark 3.
1) From Theorems 9 and 10, we can get new Hermite-Hadamard-Fejér type inequalities for co-ordinated harmonically convex functions via Riemann-Liouville fractional integral by taking $\rho_1 = \rho_2 = 1$.
2) From Theorems 9 and 10, we can get new Hermite-Hadamard-Fejér type inequalities for co-ordinated harmonically convex functions via classical integral by taking $\rho_1 = \rho_2 = 1$ and $\alpha = \beta = 1$.

4. Conclusion

In this paper, firstly we established the Hermite-Hadamard-Fejér type inequalities for harmonically convex function in one dimension which is further used to establish the Hermite-Hadamard-Fejér type inequalities for harmonically convex function via Katugampola fractional integral. The results provided in our paper are the generalizations of some earlier results.

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References

[1] Hadamard, J. (1893). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. *Journal de Mathématiques Pures et Appliquées, 58*, 171–216.
[2] Fejér, L. (1906). Uber die fourierreihen, II, Math. *Naturwiss. Anz Ungar. Akad. Wiss, 24*, 369–390. (Hungarian).
[3] Dragomir, S. S. (2011). Hermite–Hadamard's type inequalities for operator convex functions. *Applied Mathematics and Computation, 218*(3), 766–772.
[4] Chen, F. (2015). Extensions of the Hermite–Hadamard inequality for harmonically convex functions via fractional integrals. *Applied Mathematics and Computation, 268*, 121–128.
[5] İşcan, İ. (2014). Hermite–Hadamard type inequalities for harmonically convex functions. *Hacettepe Journal of Mathematics and Statistics, 43*(6), 935–942.
[6] Mehreen, N., & Anwar, M. (2018). Integral inequalities for some convex functions via generalized fractional integrals. *Journal of Inequalities and Applications, 2018*, Article No 208. DOI: https://doi.org/10.1186/s13660–018–1807–7.
[7] Mehreen, N., & Anwar, M. (2019). Hermite–Hadamard type inequalities for exponentially p–convex functions and exponentially $s-$convex functions in the second sense with applications. *Journal of Inequalities and Applications, 2019*, Article No 92. DOI: https://doi.org/10.1186/s13660–019–2047–1.
[8] Mehreen, N., & Anwar, M. (2020). Hermite–Hadamard type inequalities via exponentially $(p,h)-$convex functions. *IEEE Access, 8*, 37589–37595.
[9] Mehreen, N., & Anwar, M. (2020). On some Hermite–Hadamard type inequalities for tgs–convex functions via generalized fractional integrals. *Advances in Difference Equations, 2020*, Article No 6. DOI: https://doi.org/10.1186/s13662–019–2457–x.
[10] Mehreen, N., & Anwar, M. (2019). Hermite–Hadamard and Hermite–Hadamard–Fejer type inequalities for $p-$convex functions via new fractional conformable integral operators. *Journal of Mathematics and Computer Science, 19*(4), 230–240.
[11] Mehreen, N., & Anwar, M. (2020). Hermite–Hadamard and Hermite–Hadamard–Fejer type inequalities for $p-$convex functions via conformable fractional integrals. *Journal of Inequalities and Applications, 2020*, Article No 107. DOI: https://doi.org/10.1186/s13660–020–02363–3.
[12] Mehreen, N., & Anwar, M. (2019). Some inequalities via $\Psi-$Riemann–Liouville fractional integrals. *AIMS Mathematics, 4*(5), 1403–1415.
[13] Set, E., Iscan, I., Sarikaya, M. Z., & Özdemir, M. E. (2015). On new inequalities of Hermite–Hadamard–Fejér type for convex functions via fractional integrals. *Applied Mathematics and Computation, 259*, 875–881.
[14] Sarikaya, M. Z., Set, E., Yaldiz, H., & Basak, N. (2013). Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling, 57*(9–10), 2403–2407.
[15] Iscan, I. (2014). Hermite–Hadamard type inequalities for harmonically convex functions. *Hacettepe Journal of Mathematics and Statistics, 43*(6), 935–942.
[16] Dragomir, S. S. (2001). On the Hadamard's inequlality for convex functions on the co–ordinates in a rectangle from the plane. *Taiwanese Journal of Mathematics,5*(4), 775–788.

[17] Sarikaya, M. Z. (2014). On the Hermite–Hadamard–type inequalities for co–ordinated convex function via fractional integrals. *Integral Transforms and Special Functions, 25*(2), 134–147.

[18] Alomari, M., & Darus, M. (2008). Co–ordinated s–convex function in the first sense with some Hadamard–type inequalities. *International Journal of Contemporary Mathematical Sciences, 3*(32), 1557–1567.

[19] Alomari, M., & Darus, M. (2009). On the Hadamard's inequality for log–convex functions on the coordinates. *Journal of Inequalities and Applications, 2009*, Article No 283147. DOI: https://doi.org/10.1155/2009/283147.

[20] Mihai, M. V., Noor, M. A., Noor, K. I., & Awan, M. U. (2015). Some integral inequalities for harmonic h–convex functions involving hypergeometric functions. *Applied Mathematics and Computation, 252*, 257–262.

[21] Noor, M. A., Noor, K. I., Iftikhar, S., & Ionescu, C. (2017). Hermite–Hadamard inequalities for co–ordinated harmonic convex functions. *Politehn. Univ. Bucharest Sci. Scientific Bulletin – Series A – Applied Mathematics and Physics, 79*, 25–34.

[22] Katugampola, U. N. (2011). New approach to a generalized fractional integral. *Applied Mathematics and Computation, 218*(3), 860–865.

[23] Yaldiz, H. (2019). On Hermite–Hadamard type inequalities via katugampola fractional integrals. *TWMS Journal of Applied and Engineering Mathematics, 9*(4), 773–785.

[24] Latif, M. A., Dragomir, S. S., & Momoniat, E. (2016). Some Fejér type inequalities for harmonically–convex functions with applications to special means. *International Journal of Analysis and Applications, 13*(1), 1–14.

[25] Iscan, I., Kunt, M., & Yazici, N. (2016). Hermite–Hadamard–Fejér type inequalities for harmonically convex functions via fractional integrals. *New Trends in Mathematical Sciences, 4*(3), 239–253.

[26] Chen, F., & Wu, S. (2014). Fejér and Hermite–Hadamard type inequalities for harmonically convex functions. *Journal of Applied Mathematics, 2014*, 1–6.