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Generalized Pell-Padovan Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we investigate the generalized Pell-Padovan sequences and we deal with, in detail, four special cases, namely, Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

Keywords: Pell-Padovan numbers; Pell-Perrin numbers; third order Fibonacci-Pell numbers; third order Lucas-Pell numbers.

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1 INTRODUCTION

The aim of this paper is to define and to explore some of the properties of generalized Pell-Padovan numbers and is to investigate, in details, four particular case, namely sequences of Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell. Before, we recall the generalized Tribonacci sequence and its some properties.

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

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$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [1,2,3,4,5,6,7,8,9,10,11,12,13].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.2}$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B \\ \beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\ \gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\ \Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \ \ \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3) \end{aligned}$$

Note that we have the following identities

$$\begin{aligned} \alpha + \beta + \gamma &= r, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

If $\Delta(r, s, t) > 0$, then the Equ. (1.2) has one real (α) and two non-real solutions with the latter being conjugate complex (in our case all roots are reals). So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{b_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{b_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{1.3}$$

where

$$b_1 = W_2 - (\beta + \gamma)W_1 + \beta\gamma W_0, \ b_2 = W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0, \ b_3 = W_2 - (\alpha + \beta)W_1 + \alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, for a proof of this result see [14]. This result of Howard and Saidak [14] is even true in the case of higher-order recurrence relations.

In this paper we consider the case $r = 0, \ s = 2, t = 1$ and in this case we write $V_n = W_n$. A generalized Pell-Padovan sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = 2V_{n-2} + V_{n-3} \tag{1.4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -2V_{-(n-1)} + V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.4) holds for all integer $n.$

(1.3) can be used to obtain Binet formula of generalized Pell-Padovan numbers. Binet formula of generalized padovan numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \quad (1.5)$$

Here, α, β and γ are the roots of the cubic equation $x^3 - 2x - 1 = 0$. Moreover

$$\begin{aligned} \alpha &= \frac{1+\sqrt{5}}{2}, \\ \beta &= \frac{1-\sqrt{5}}{2}, \\ \gamma &= -1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma &= 0, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -2, \\ \alpha\beta\gamma &= 1. \end{aligned}$$

The first few generalized Pell-Padovan numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Pell-Padovan numbers

n	V_n	V_{-n}
0	V_0	$\ldots$
1	V_1	$V_2 - 2V_0$
2	V_2	$-2V_2 + V_1 + 4V_0$
3	$2V_1 + V_0$	$4V_2 - 2V_1 - 7V_0$
4	$2V_2 + V_1$	$-7V_2 + 4V_1 + 12V_0$
5	$V_2 + 4V_1 + 2V_0$	$12V_2 - 7V_1 - 20V_0$
6	$4V_2 + 4V_1 + V_0$	$-20V_2 + 12V_1 + 33V_0$
7	$4V_2 + 9V_1 + 4V_0$	$33V_2 - 20V_1 - 54V_0$
8	$9V_2 + 12V_1 + 4V_0$	$-54V_2 + 33V_1 + 88V_0$
9	$12V_2 + 22V_1 + 9V_0$	$88V_2 - 54V_1 - 143V_0$
10	$1022V_2 + 33V_1 + 12V_0$	$-143V_2 + 88V_1 + 232V_0$

Now we define four special cases of the sequence $\{V_n\}$. Pell-Padovan sequence$\{R_n\}_{n\geq 0}$, Pell-Perrin sequence $\{C_n\}_{n\geq 0}$, third order Fibonacci-Pell sequence $\{G_n\}_{n\geq 0}$ and third order Lucas-Pell sequence $\{B_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{array}{rcll} R_{n+3} & = & R_{n+1}+R_n, & R_0=1, R_1=1, R_2=1, \\ C_{n+3} & = & C_{n+1}+C_n, & C_0=3, C_1=0, C_2=2, \\ G_{n+3} & = & G_{n+1}+G_n, & G_0=1, G_1=0, G_2=2, \\ B_{n+3} & = & B_{n+1}+B_n, & B_0=3, B_1=0, B_2=4. \end{array}$$

The sequences $\{R_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, $\{G_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$R_{-n} = -2R_{-(n-1)} + R_{-(n-3)} \tag{1.6}$$
$$C_{-n} = -2C_{-(n-1)} + C_{-(n-3)} \tag{1.7}$$
$$G_{-n} = -2G_{-(n-1)} + G_{-(n-3)} \tag{1.8}$$
$$B_{-n} = -2B_{-(n-1)} + B_{-(n-3)} \tag{1.9}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.6), (1.7), (1.8) and (1.9) hold for all integer n.

For more information on Pell-Padovan sequence, see [15,16,17,18,19,20,21,22].

Next, we present the first few values of the Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
R_n	1	1	1	3	3	7	9	17	25	43	67	111	177	289
R_{-n}		-1	3	-5	9	-15	25	-41	67	-109	177	-287	465	-753
C_n	3	0	2	3	4	8	11	20	30	51	80	132	211	344
C_{-n}		-4	8	-13	22	-36	59	-96	156	-253	410	-664	1075	-1740
G_n	1	0	2	1	4	4	9	12	22	33	56	88	145	232
G_{-n}		0	0	1	-2	4	-7	12	-20	33	-54	88	-143	232
B_n	3	0	4	3	8	10	19	28	48	75	124	198	323	520
B_{-n}		-2	4	-5	8	-12	19	-30	48	-77	124	-200	323	-522

For all integers n, Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell numbers (using initial conditions in (1.5)) can be expressed using Binet's formulas as

$$\begin{array}{rcl} R_n & = & (1-\dfrac{1}{\sqrt{5}})\alpha^n + (1+\dfrac{1}{\sqrt{5}})\beta^n - \gamma^n, \\ C_n & = & (2-\dfrac{3}{\sqrt{5}})\alpha^n + (2+\dfrac{3}{\sqrt{5}})\beta^n - \gamma^n, \\ G_n & = & \dfrac{1}{\sqrt{5}}\alpha^n - \dfrac{1}{\sqrt{5}}\beta^n + \gamma^n, \\ B_n & = & \alpha^n + \beta^n + \gamma^n, \end{array}$$

respectively.

R_n is the sequence A066983 in [23] associated with the relation

$$R_{n+2} = R_{n+1} + R_n + (-1)^n, \text{ with } R_1 = R_2 = 1.$$

C_n is not indexed in [23].

G_n is the sequence A008346 in [23] associated with the relation

$$G_n = F_n + (-1)^n$$

where F_n is Fibonacci sequence which is given as

$$F_n = F_{n-1} + F_{n-2} \text{ with } F_0 = 0 \text{ and } F_1 = 1.$$

B_n is the sequence A099925 in [23] associated with the relation

$$B_n = L_n + (-1)^n$$

where L_n is Lucas sequence which is given as

$$L_n = L_{n-1} + L_{n-2} \text{ with } L_0 = 2 \text{ and } L_1 = 1.$$

Since

$$F_{-n} = (-1)^{n+1}F_n \text{ and } L_{-n} = (-1)^n L_n$$

we get

$$G_{-n} = (-1)^{n+1}G_n + 1 + (-1)^n = (-1)^n(1 - F_n)$$

and

$$B_{-n} = (-1)^n B_n - 1 + (-1)^n = (-1)^n(L_n + 1).$$

2 GENERATING FUNCTIONS

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Pell-Padovan sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + V_1 x + (V_2 - 2V_0)x^2}{1 - 2x^2 - x^3}. \tag{2.1}$$

Proof. Using the definition of generalized Pell-Padovan numbers, and substracting $2x^2 \sum_{n=0}^{\infty} V_n x^n$ and $x^3 \sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
(1 - 2x^2 - x^3)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - 2x^2 \sum_{n=0}^{\infty} V_n x^n - x^3 \sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3} \\
&= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=2}^{\infty} V_{n-2} x^n - \sum_{n=3}^{\infty} V_{n-3} x^n \\
&= (V_0 + V_1 x + V_2 x^2) - 2V_0 x^2 + \sum_{n=3}^{\infty}(V_n - 2V_{n-2} - V_{n-3})x^n \\
&= V_0 + V_1 x + V_2 x^2 - 2V_0 x^2 \\
&= V_0 + V_1 x + (V_2 - 2V_0)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + V_1 x + (V_2 - 2V_0)x^2}{1 - 2x^2 - x^3}.$$

The previous lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} R_n x^n &= \frac{-x^2 + x + 1}{1 - 2x^2 - x^3}, \\
\sum_{n=0}^{\infty} C_n x^n &= \frac{3 - 4x^2}{1 - 2x^2 - x^3}, \\
\sum_{n=0}^{\infty} G_n x^n &= \frac{1}{1 - 2x^2 - x^3}, \\
\sum_{n=0}^{\infty} B_n x^n &= \frac{3 - 2x^2}{1 - 2x^2 - x^3},
\end{aligned}$$

respectively.

3 OBTAINING BINET FORMULA FROM GENERATING FUNCTION

We next find Binet formula of generalized Pell-Padovan numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3.1. *(Binet formula of generalized Pell-Padovan numbers)*

$$V_n = \frac{d_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{d_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{d_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{3.1}$$

where

$$\begin{aligned}
d_1 &= V_0 \alpha^2 + V_1 \alpha + (V_2 - 2V_0), \\
d_2 &= V_0 \beta^2 + V_1 \beta + (V_2 - 2V_0), \\
d_3 &= V_0 \gamma^2 + V_1 \gamma + (V_2 - 2V_0).
\end{aligned}$$

Proof. Let

$$h(x) = 1 - 2x^2 - x^3.$$

Then for some α, β and γ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)$$

i.e.,

$$1 - 2x^2 - x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x) \tag{3.2}$$

Hence $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h\left(\frac{1}{x}\right) = 1 - \frac{2}{x^2} - \frac{1}{x^3} = 0.$$

This implies $x^3 - 2x - 1 = 0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + V_1 x + (V_2 - 2V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{V_0 + V_1 x + (V_2 - 2V_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \quad (3.3)$$

So

$$V_0 + V_1 x + (V_2 - 2V_0)x^2 = A_1(1-\beta x)(1-\gamma x) + A_2(1-\alpha x)(1-\gamma x) + A_3(1-\alpha x)(1-\beta x).$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + V_1\frac{1}{\alpha} + (V_2 - 2V_0)\frac{1}{\alpha^2} = A_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})$. This gives

$$A_1 = \frac{\alpha^2(V_0 + V_1\frac{1}{\alpha} + (V_2 - 2V_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)} = \frac{V_0\alpha^2 + V_1\alpha + (V_2 - 2V_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2 = \frac{V_0\beta^2 + V_1\beta + (V_2 - 2V_0)}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{V_0\gamma^2 + V_1\gamma + (V_2 - 2V_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1\sum_{n=0}^{\infty}\alpha^n x^n + A_2\sum_{n=0}^{\infty}\beta^n x^n + A_3\sum_{n=0}^{\infty}\gamma^n x^n = \sum_{n=0}^{\infty}(A_1\alpha^n + A_2\beta^n + A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where

$$\begin{aligned} A_1 &= \frac{V_0\alpha^2 + V_1\alpha + (V_2 - 2V_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{V_0\beta^2 + V_1\beta + (V_2 - 2V_0)}{(\beta-\alpha)(\beta-\gamma)} \\ A_3 &= \frac{V_0\gamma^2 + V_1\gamma + (V_2 - 2V_0)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

and then we get (3.1).

Note that from (1.5) and (3.1) we have

$$\begin{aligned} V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0 &= V_0\alpha^2 + V_1\alpha + (V_2 - 2V_0), \\ V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0 &= V_0\beta^2 + V_1\beta + (V_2 - 2V_0), \\ V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0 &= V_0\gamma^2 + V_1\gamma + (V_2 - 2V_0). \end{aligned}$$

Next, using Theorem 3.1, we present the Binet formulas of Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell sequences.

Corollary 3.2. *Binet formulas of Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell sequences are*

$$\begin{aligned}
R_n &= (1-\frac{1}{\sqrt{5}})\alpha^n+(1+\frac{1}{\sqrt{5}})\beta^n-\gamma^n,\\
C_n &= (2-\frac{3}{\sqrt{5}})\alpha^n+(2+\frac{3}{\sqrt{5}})\beta^n-\gamma^n,\\
G_n &= \frac{1}{\sqrt{5}}\alpha^n-\frac{1}{\sqrt{5}}\beta^n+\gamma^n,\\
B_n &= \alpha^n+\beta^n+\gamma^n,
\end{aligned}$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [24]. Take $k=i=3$ in Corollary 3.1 in [24]. Let

$$\begin{aligned}
\Lambda &= \begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{pmatrix},\\
\Lambda_2 &= \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}.
\end{aligned}$$

Then the Binet formula for Pell-Padovan numbers is

$$\begin{aligned}
R_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{3} R_{4-j}\det(\Lambda_j) = \frac{1}{\Lambda}(R_3\det(\Lambda_1)+R_2\det(\Lambda_2)+R_1\det(\Lambda_3))\\
&= \frac{1}{\det(\Lambda)}(3\det(\Lambda_1)+\det(\Lambda_2)+\det(\Lambda_3))\\
&= \left(3\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}.
\end{aligned}$$

Similarly, we obtain the Binet formula for Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell as

$$\begin{aligned}
C_n &= \frac{1}{\Lambda}(C_3\det(\Lambda_1)+C_2\det(\Lambda_2)+C_1\det(\Lambda_3))\\
&= \left(3\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 2\begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}.
\end{aligned}$$

and

$$\begin{aligned}
G_n &= \frac{1}{\Lambda}(G_3\det(\Lambda_1)+G_2\det(\Lambda_2)+G_1\det(\Lambda_3))\\
&= \left(\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 2\begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix}\right) \Big/ \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}.
\end{aligned}$$

and

$$\begin{aligned} B_n &= \frac{1}{\Lambda}(B_3\det(\Lambda_1)+B_2\det(\Lambda_2)+B_1\det(\Lambda_3)) \\ &= \left(3\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 4\begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix}\right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \end{aligned}$$

respectively.

4 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Pell-Padovan sequence $\{V_n\}_{n\geq 0}$.

Theorem 4.1 (Simson Formula of Generalized Pell-Padovan Numbers). *For all integers n, we have*

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. (4.1) is given in Soykan [25].

The previous theorem gives the following results as particular examples.

Corollary 4.2. *For all integers n, Simson formula of Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell numbers are given as*

$$\begin{vmatrix} R_{n+2} & R_{n+1} & R_n \\ R_{n+1} & R_n & R_{n-1} \\ R_n & R_{n-1} & R_{n-2} \end{vmatrix} = -4$$

and

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = -11$$

and

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -1$$

and

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = 5$$

respectively.

5 SOME IDENTITIES

In this section, we obtain some identities of Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell numbers. First, we can give a few basic relations between $\{R_n\}$ and $\{C_n\}$.

Lemma 5.1. *The following equalities are true:*

$$\begin{aligned} 2C_n &= -12R_{n+4}+7R_{n+3}+21R_{n+2}, \\ 2C_n &= 7R_{n+3}-3R_{n+2}-12R_{n+1}, \\ 2C_n &= -3R_{n+2}+2R_{n+1}+7R_n, \\ 2C_n &= 2R_{n+1}+R_n-3R_{n-1}, \\ 2C_n &= R_n+R_{n-1}+2R_{n-2}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} 11R_n &= -3C_{n+4}-C_{n+3}+13C_{n+2}, \\ 11R_n &= -C_{n+3}+7C_{n+2}-3C_{n+1}, \\ 11R_n &= 7C_{n+2}-5C_{n+1}-C_n, \\ 11R_n &= -5C_{n+1}+13C_n+7C_{n-1}, \\ 11R_n &= 13C_n-3C_{n-1}-5C_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.1). To show (5.1), writing

$$C_n = a\times R_{n+4}+b\times R_{n+3}+c\times R_{n+2}$$

and solving the system of equations

$$\begin{aligned} C_0 &= a\times R_4+b\times R_3+c\times R_2 \\ C_1 &= a\times R_5+b\times R_4+c\times R_3 \\ C_2 &= a\times R_6+b\times R_5+c\times R_4 \end{aligned}$$

we find that $a=-6, b=\frac{7}{2}, c=\frac{21}{2}$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{R_n\}$ and $\{G_n\}$.

Lemma 5.2. *The following equalities are true:*

$$\begin{aligned} 2G_n &= R_{n+3}-R_{n+2}, \\ 2G_n &= -R_{n+2}+2R_{n+1}+R_n, \\ 2G_n &= 2R_{n+1}-R_n-R_{n-1}, \\ 2G_n &= -R_n+3R_{n-1}+2R_{n-2}, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} R_n &= 9G_{n+4}-5G_{n+3}-15G_{n+2}, \\ R_n &= -5G_{n+3}+3G_{n+2}+9G_{n+1}, \\ R_n &= 3G_{n+2}-G_{n+1}-5G_n, \\ R_n &= -G_{n+1}+G_n+3G_{n-1}, \\ R_n &= G_n+G_{n-1}-G_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.2). To show (5.2), writing

$$G_n = a \times R_{n+4} + b \times R_{n+3} + c \times R_{n+2}$$

and solving the system of equations

$$\begin{aligned} G_0 &= a \times R_4 + b \times R_3 + c \times R_2 \\ G_1 &= a \times R_5 + b \times R_4 + c \times R_3 \\ G_2 &= a \times R_6 + b \times R_5 + c \times R_4 \end{aligned}$$

we find that $a = 0, b = \frac{1}{2}, c = -\frac{1}{2}$. The other equalities can be proved similarly.

Now, we give a few basic relations between $\{R_n\}$ and $\{B_n\}$.

Lemma 5.3. *The following equalities are true:*

$$\begin{aligned} 2B_n &= -6R_{n+4} + 5R_{n+3} + 9R_{n+2}, \\ 2B_n &= 5R_{n+3} - 3R_{n+2} - 6R_{n+1}, \\ 2B_n &= -3R_{n+2} + 4R_{n+1} + 5R_n, \\ 2B_n &= 4R_{n+1} - R_n - 3R_{n-1}, \\ 2B_n &= -R_n + 5R_{n-1} + 4R_{n-2}, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} 5R_n &= -27B_{n+4} + 19B_{n+3} + 41B_{n+2}, \\ 5R_n &= 19B_{n+3} - 13B_{n+2} - 27B_{n+1}, \\ 5R_n &= -13B_{n+2} + 11B_{n+1} + 19B_n, \\ 5R_n &= 11B_{n+1} - 7B_n - 13B_{n-1}, \\ 5R_n &= -7B_n + 9B_{n-1} + 11B_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.3). To show (5.3), writing

$$B_n = a \times R_{n+4} + b \times R_{n+3} + c \times R_{n+2}$$

and solving the system of equations

$$\begin{aligned} B_0 &= a \times R_4 + b \times R_3 + c \times R_2 \\ B_1 &= a \times R_5 + b \times R_4 + c \times R_3 \\ B_2 &= a \times R_6 + b \times R_5 + c \times R_4 \end{aligned}$$

we find that $a = -3, b = \frac{5}{2}, c = \frac{9}{2}$. The other equalities can be proved similarly.

Next, we present a few basic relations between $\{C_n\}$ and $\{G_n\}$.

Lemma 5.4. *The following equalities are true*

$$\begin{aligned} 11G_n &= -6C_{n+4} + 9C_{n+3} + 4C_{n+2}, \\ 11G_n &= 9C_{n+3} - 8C_{n+2} - 6C_{n+1}, \\ 11G_n &= -8 \times C_{n+2} + 12C_{n+1} + 9C_n, \\ 11G_n &= 12 \times C_{n+1} - 7C_n - 8C_{n-1}, \\ 11G_n &= -7C_n + 16C_{n-1} + 12C_{n-2}, \end{aligned} \tag{5.4}$$

and

$$\begin{array}{rcl} C_n & = & 22G_{n+4} - 13G_{n+3} - 36G_{n+2}, \\ C_n & = & -13G_{n+3} + 8G_{n+2} + 22G_{n+1}, \\ C_n & = & 8G_{n+2} - 4G_{n+1} - 13G_n, \\ C_n & = & -4G_{n+1} + 3G_n + 8G_{n-1}, \\ C_n & = & 3G_n - 4G_{n-2}. \end{array}$$

Proof. Note that all the identities hold for all integers n. We prove (5.4). To show (5.4), writing

$$G_n = a \times C_{n+4} + b \times C_{n+3} + c \times C_{n+2}$$

and solving the system of equations

$$\begin{array}{rcl} G_0 & = & a \times C_4 + b \times C_3 + c \times C_2 \\ G_1 & = & a \times C_5 + b \times C_4 + c \times C_3 \\ G_2 & = & a \times C_6 + b \times C_5 + c \times C_4 \end{array}$$

we find that $a = -\frac{6}{11}, b = \frac{9}{11}, c = \frac{4}{11}$. The other equalities can be proved similarly.

Next, we give a few basic relations between $\{B_n\}$ and $\{C_n\}$.

Lemma 5.5. *The following equalities are true*

$$\begin{array}{rcl} 5C_n & = & -56B_{n+4} + 37B_{n+3} + 88B_{n+2}, \\ 5C_n & = & 37B_{n+3} - 24B_{n+2} - 56B_{n+1}, \\ 5C_n & = & -24B_{n+2} + 18B_{n+1} + 37B_n, \\ 5C_n & = & 18B_{n+1} - 11B_n - 24B_{n-1}, \\ 3C_n & = & -11B_n + 12B_{n-1} + 18B_{n-2}, \end{array} \tag{5.5}$$

and

$$\begin{array}{rcl} 11B_n & = & -20C_{n+4} + 19C_{n+3} + 28C_{n+2}, \\ 11B_n & = & 19C_{n+3} - 12C_{n+2} - 20C_{n+1}, \\ 11B_n & = & -12C_{n+2} + 18C_{n+1} + 19C_n, \\ 11B_n & = & 18C_{n+1} - 5C_n - 12C_{n-1}, \\ 11B_n & = & -5C_n + 24 \times C_{n-1} + 18C_{n-2}. \end{array}$$

Proof. Note that all the identities hold for all integers n. We prove (5.5). To show (5.5), writing

$$C_n = a \times B_{n+4} + b \times B_{n+3} + c \times B_{n+2}$$

and solving the system of equations

$$\begin{array}{rcl} C_0 & = & a \times B_4 + b \times B_3 + c \times B_2 \\ C_1 & = & a \times B_5 + b \times B_4 + c \times B_3 \\ C_2 & = & a \times B_6 + b \times B_5 + c \times B_4 \end{array}$$

we find that $a = -\frac{56}{5}, b = \frac{37}{5}, c = \frac{88}{5}$. The other equalities can be proved similarly.

Now, we present a few basic relations between $\{G_n\}$ and $\{B_n\}$.

Lemma 5.6. *The following equalities are true*

$$\begin{aligned} B_n &= 8G_{n+4} - 5G_{n+3} - 12G_{n+2}, \\ B_n &= -5G_{n+3} + 4G_{n+2} + 8G_{n+1}, \\ B_n &= 4G_{n+2} - 2G_{n+1} - 5G_n, \\ B_n &= -2G_{n+1} + 3G_n + 4G_{n-1}, \\ B_n &= 3G_n - 2G_{n-2}, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} 5G_n &= 12B_{n+4} - 9B_{n+3} - 16B_{n+2}, \\ 5G_n &= -9B_{n+3} + 8B_{n+2} + 12B_{n+1}, \\ 5G_n &= 8B_{n+2} - 6B_{n+1} - 9B_n, \\ 5G_n &= -6B_{n+1} + 7B_n + 8B_{n-1}, \\ 5G_n &= 7B_n - 4B_{n-1} - 6B_{n-2}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (5.6). To show (5.6), writing

$$B_n = a \times G_{n+4} + b \times G_{n+3} + c \times G_{n+2}$$

and solving the system of equations

$$\begin{aligned} B_0 &= a \times G_4 + b \times G_3 + c \times G_2 \\ B_1 &= a \times G_5 + b \times G_4 + c \times G_3 \\ B_2 &= a \times G_6 + b \times G_5 + c \times G_4 \end{aligned}$$

we find that $a = 8, b = -5, c = -12$. The other equalities can be proved similarly.

6 LINEAR SUMS

The following Theorem presents some formulas of generalized Pell-Padovan numbers with positive subscripts.

Theorem 6.1. *If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = \frac{1}{2}\left(V_{n+3} + V_{n+2} - V_{n+1} - V_2 - V_1 + V_0\right).$

(b) $\sum_{k=0}^{n} V_{2k} = V_{2n+1} + (V_2 - V_1 - V_0)\, n + V_0 - V_1.$

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{2}\left(V_{2n+3} + V_{2n+2} - V_{2n+1} + 2n\left(-V_2 + V_1 + V_0\right) - V_2 + V_1 - V_0\right).$

The above theorem is given in [26, Theorem 2.13].

From the above Theorem we have the following Corollary which gives sum formulas of Pell-Padovan numbers (take $V_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 6.2. *For $n \geq 0$, Pell-Padovan numbers have the following property:*

(a) $\sum_{k=0}^{n} R_k = \frac{1}{2}\left(R_{n+3} + R_{n+2} - R_{n+1} - 1\right).$

(b) $\sum_{k=0}^{n} R_{2k} = R_{2n+1} - n.$

(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{2}\left(R_{2n+3} + R_{2n+2} - R_{2n+1} + 2n - 1\right).$

Proof.

(a) Take $R_0 = 1, R_1 = 1, R_2 = 1$ in the following sum formula

$$\sum_{k=0}^{n} R_k = \frac{1}{2}\left(R_{n+3} + R_{n+2} - R_{n+1} - R_2 - R_1 + R_0\right).$$

(b) Take $R_0 = 1, R_1 = 1, R_2 = 1$ in the following sum formula

$$\sum_{k=0}^{n} R_{2k} = R_{2n+1} + \left(R_2 - R_1 - R_0\right) n + R_0 - R_1.$$

(c) Take $R_0 = 1, R_1 = 1, R_2 = 1$ in the following sum formula

$$\sum_{k=0}^{n} R_{2k+1} = \frac{1}{2}\left(R_{2n+3} + R_{2n+2} - R_{2n+1} + 2n\left(-R_2 + R_1 + R_0\right) - R_2 + R_1 - R_0\right).$$

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last Theorem, we have the following Corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 6.3. *For $n \geq 0$, Pell-Perrin numbers have the following property:*

(a) $\sum_{k=0}^{n} C_k = \frac{1}{2}\left(C_{n+3} + C_{n+2} - C_{n+1} + 1\right).$

(b) $\sum_{k=0}^{n} C_{2k} = C_{2n+1} - n + 3.$

(c) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{2}\left(C_{2n+3} + C_{2n+2} - C_{2n+1} + 2n - 5\right).$

From the last Theorem, we have the following Corollary which gives linear sum formulas of third order Fibonacci-Pell numbers (take $V_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 6.4. *For $n \geq 0$, third order Fibonacci-Pell numbers have the following property:*

(a) $\sum_{k=0}^{n} G_k = \frac{1}{2}\left(G_{n+3} + G_{n+2} - G_{n+1} - 1\right).$

(b) $\sum_{k=0}^{n} G_{2k} = G_{2n+1} + n + 1.$

(c) $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{2}\left(G_{2n+3} + G_{2n+2} - G_{2n+1} - 2n - 3\right).$

From the last Theorem, we have the following Corollary which gives linear sum formulas of third order Lucas-Pell numbers (take $V_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$).

Corollary 6.5. *For $n \geq 0$, third order Lucas-Pell numbers have the following property:*

(a) $\sum_{k=0}^{n} B_k = \frac{1}{2}\left(B_{n+3} + B_{n+2} - B_{n+1} - 1\right).$

(b) $\sum_{k=0}^{n} B_{2k} = B_{2n+1} + n + 3.$

(c) $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{2}\left(B_{2n+3} + B_{2n+2} - B_{2n+1} - 2n - 7\right).$

The following theorem presents some formulas of generalized Pell-Padovan numbers with negative subscripts.

Theorem 6.6. *If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{2}\left(-3V_{-n-1} - 3V_{-n-2} - V_{-n-3} + V_2 + V_1 - V_0\right).$

(b) $\sum_{k=1}^{n} V_{-2k} = -V_{-2n+1} + V_{-2n} + (V_1 - V_0) + (V_2 - V_1 - V_0)n.$

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{2}(V_{-2n+1} - 3V_{-2n} - V_{-2n-1} + (V_2 - V_1 + V_0) + 2(-V_2 + V_1 + V_0)n).$

The above theorem is given in [26, Theorem 3.13].

From the last theorem, we have the following corollary which gives sum formula of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R = 1, R_2 = 1$).

Corollary 6.7. *For $n \geq 1,$ Pell-Padovan numbers have the following property:*

(a) $\sum_{k=1}^{n} R_{-k} = \frac{1}{2}\left(-3R_{-n-1} - 3R_{-n-2} - R_{-n-3} + 1\right).$

(b) $\sum_{k=1}^{n} R_{-2k} = -R_{-2n+1} + R_{-2n} - n.$

(c) $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{2}(R_{-2n+1} - 3R_{-2n} - R_{-2n-1} + 1 + 2n).$

Proof.

(a) Take $R_0 = 1, R_1 = 1, R_2 = 1$ in the following sum formula

$$\sum_{k=1}^{n} R_{-k} = \frac{1}{2}\left(-3R_{-n-1} - 3R_{-n-2} - R_{-n-3} + R_2 + R_1 - R_0\right).$$

(b) Take $R_0 = 1, R_1 = 1, R_2 = 1$ in the following sum formula

$$\sum_{k=1}^{n} R_{-2k} = -R_{-2n+1} + R_{-2n} + (R_1 - R_0) + (R_2 - R_1 - R_0)n.$$

(c) Take $R_0 = 1, R_1 = 1, R_2 = 1$ in the following sum formula

$$\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{2}(R_{-2n+1} - 3R_{-2n} - R_{-2n-1} + (R_2 - R_1 + R_0) + 2(-R_2 + R_1 + R_0)n).$$

Taking $W_n = C_n$ with $C_0 = 3, C = 0, C_2 = 2$ in the last theorem, we have the following corollary which gives sum formulas of Pell-Perrin numbers.

Corollary 6.8. *For $n \geq 1,$ Pell-Perrin numbers have the following property:*

(a) $\sum_{k=1}^{n} C_{-k} = \frac{1}{2}\left(-3C_{-n-1} - 3C_{-n-2} - C_{-n-3} - 1\right).$

(b) $\sum_{k=1}^{n} C_{-2k} = -C_{-2n+1} + C_{-2n} - 3 - n.$

(c) $\sum_{k=1}^{n} C_{-2k+1} = \frac{1}{2}(C_{-2n+1} - 3C_{-2n} - C_{-2n-1} + 5 + 2n).$

From the last theorem, we have the following corollary which gives sum formula of third order Fibonacci-Pell numbers (take $V_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 6.9. *For $n \geq 1,$ third order Fibonacci-Pell numbers have the following property:*

(a) $\sum_{k=1}^{n} G_{-k} = \frac{1}{2}\left(-3G_{-n-1} - 3G_{-n-2} - G_{-n-3} + 1\right)$

(b) $\sum_{k=1}^{n} G_{-2k} = -G_{-2n+1} + G_{-2n} - 1 + n.$

(c) $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{2}(G_{-2n+1} - 3G_{-2n} - G_{-2n-1} + 3 - 2n)$

Taking $_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last theorem, we have the following corollary which gives sum formulas of third order Lucas-Pell numbers.

Corollary 6.10. *For $n \geq 1,$ third order Lucas-Pell numbers have the following property:*

(a) $\sum_{k=1}^{n} B_{-k} = \frac{1}{2}\left(-3B_{-n-1} - 3B_{-n-2} - B_{-n-3} + 1\right).$

(b) $\sum_{k=1}^{n} B_{-2k} = -B_{-2n+1} + B_{-2n} - 3 + n.$

(c) $\sum_{k=1}^{n} B_{-2k+1} = \frac{1}{2}(B_{-2n+1} - 3B_{-2n} - B_{-2n-1} + 7 - 2n).$

7 MATRICES RELATED WITH GENERALIZED PELL-PADOVAN NUMBERS

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [27]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1.$ From (1.4) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = R$ in (7.2) we have

$$\begin{pmatrix} R_{n+2} \\ R_{n+1} \\ R_n \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} R_{n+1} \\ R_n \\ R_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$\begin{aligned} B_n &= \begin{pmatrix} \frac{1}{2}(R_{n+3} - R_{n+2}) & \frac{1}{2}(R_{n+4} - R_{n+3}) & \frac{1}{2}(R_{n+2} - R_{n+1}) \\ \frac{1}{2}(R_{n+2} - R_{n+1}) & \frac{1}{2}(R_{n+3} - R_{n+2}) & \frac{1}{2}(R_{n+1} - R_n) \\ \frac{1}{2}(R_{n+1} - R_n) & \frac{1}{2}(R_{n+2} - R_{n+1}) & \frac{1}{2}(R_n - R_{n-1}) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} R_{n+3} - R_{n+2} & R_{n+4} - R_{n+3} & R_{n+2} - R_{n+1} \\ R_{n+2} - R_{n+1} & R_{n+3} - R_{n+2} & R_{n+1} - R_n \\ R_{n+1} - R_n & R_{n+2} - R_{n+1} & R_n - R_{n-1} \end{pmatrix} \end{aligned}$$

and

$$D_n = \begin{pmatrix} \frac{1}{2}(V_{n+3} - V_{n+2}) & \frac{1}{2}(V_{n+4} - V_{n+3}) & \frac{1}{2}(V_{n+2} - V_{n+1}) \\ \frac{1}{2}(V_{n+2} - V_{n+1}) & \frac{1}{2}(V_{n+3} - V_{n+2}) & \frac{1}{2}(V_{n+1} - V_n) \\ \frac{1}{2}(V_{n+1} - V_n) & \frac{1}{2}(V_{n+2} - V_{n+1}) & \frac{1}{2}(V_n - V_{n-1}) \end{pmatrix}.$$

Theorem 7.1. *For all integer $m, n \geq 0,$ we have*

(a) $B_n = A^n$

(b) $D_1 A^n = A^n D_1$

(c) $D_{n+m} = D_n B_m = B_m D_n$.

Proof.

(a) By expanding the vectors on the both sides of (7.3) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

NOTE: (a) can be proved by mathematical induction (using directly).

(b) Using (a) and definition of D_1, (b) follows.

(c) We have

$$\begin{aligned} AD_{n-1} &= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(V_{n+2} - V_{n+1}) & \frac{1}{2}(V_{n+3} - V_{n+2}) & \frac{1}{2}(V_{n+1} - V_n) \\ \frac{1}{2}(V_{n+1} - V_n) & \frac{1}{2}(V_{n+2} - V_{n+1}) & \frac{1}{2}(V_n - V_{n-1}) \\ \frac{1}{2}(V_n - V_{n-1}) & \frac{1}{2}(V_{n+1} - V_n) & \frac{1}{2}(V_{n-1} - V_{n-2}) \end{pmatrix} \\ &= \begin{pmatrix} V_{n+1} - \frac{1}{2}V_{n-1} - \frac{1}{2}V_n & V_{n+2} - \frac{1}{2}V_{n+1} - \frac{1}{2}V_n & V_n - \frac{1}{2}V_{n-1} - \frac{1}{2}V_{n-2} \\ \frac{1}{2}V_{n+2} - \frac{1}{2}V_{n+1} & \frac{1}{2}V_{n+3} - \frac{1}{2}V_{n+2} & \frac{1}{2}V_{n+1} - \frac{1}{2}V_n \\ \frac{1}{2}V_{n+1} - \frac{1}{2}V_n & \frac{1}{2}V_{n+2} - \frac{1}{2}V_{n+1} & \frac{1}{2}V_n - \frac{1}{2}V_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(V_{n+3} - V_{n+2}) & \frac{1}{2}(V_{n+4} - V_{n+3}) & \frac{1}{2}(V_{n+2} - V_{n+1}) \\ \frac{1}{2}(V_{n+2} - V_{n+1}) & \frac{1}{2}(V_{n+3} - V_{n+2}) & \frac{1}{2}(V_{n+1} - V_n) \\ \frac{1}{2}(V_{n+1} - V_n) & \frac{1}{2}(V_{n+2} - V_{n+1}) & \frac{1}{2}(V_n - V_{n-1}) \end{pmatrix} = D_n, \end{aligned}$$

i.e. $D_n = AD_{n-1}$. From the last equation, using induction we obtain $D_n = A^{n-1}D_1$. Now

$$D_{n+m} = A^{n+m-1}D_1 = A^{n-1}A^m D_1 = A^{n-1}D_1 A^m = D_n B_m$$

and similarly

$$D_{n+m} = B_m D_n.$$

Note that Theorem 7.1 is true for all integers m, n.

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-2} + A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 1$$

for all integer m and n.

Theorem 7.2. *For $m, n \geq 0$ we have*

$$\begin{aligned} 2(V_{n+m+2} - V_{n+m+1}) &= (V_{n+3} - V_{n+2})(R_{m+2} - R_{m+1}) \\ &\quad + (V_{n+2} - V_{n+1})(R_{m+3} - R_{m+2}) + (V_{n+1} - V_n)(R_{m+1} - R_m). \end{aligned} \tag{7.4}$$

Proof. From the equation $D_{n+m} = D_n B_m = B_m D_n$ we see that an element of D_{n+m} is the product of row D_n and a column B_m. From the last equation we say that an element of D_{n+m} is the product of a row D_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices D_{n+m} and $D_n B_m$. This completes the proof.

Remark 7.1. By induction, it can be proved that for all integers $m,n \leq 0$, (7.4) holds. So for all integers m,n, (7.4) is true.

Corollary 7.3. *For all integers* m,n, *we have*

$$\begin{aligned}
2(R_{n+m+2}-R_{n+m+1}) &= (R_{n+3}-R_{n+2})(R_{m+2}-R_{m+1}) \\
&\quad +(R_{n+2}-R_{n+1})(R_{m+3}-R_{m+2})+(R_{n+1}-R_{n})(R_{m+1}-R_{m}), \\
2(C_{n+m+2}-C_{n+m+1}) &= (C_{n+3}-C_{n+2})(R_{m+2}-R_{m+1}) \\
&\quad +(C_{n+2}-C_{n+1})(R_{m+3}-R_{m+2})+(C_{n+1}-C_{n})(R_{m+1}-R_{m}), \\
2(G_{n+m+2}-G_{n+m+1}) &= (G_{n+3}-G_{n+2})(R_{m+2}-R_{m+1}) \\
&\quad +(G_{n+2}-G_{n+1})(R_{m+3}-R_{m+2})+(G_{n+1}-G_{n})(R_{m+1}-R_{m}), \\
2(B_{n+m+2}-B_{n+m+1}) &= (B_{n+3}-B_{n+2})(R_{m+2}-R_{m+1}) \\
&\quad +(B_{n+2}-B_{n+1})(R_{m+3}-R_{m+2})+(B_{n+1}-B_{n})(R_{m+1}-R_{m}).
\end{aligned}$$

Note that using Theorem 7.1 (a) and the property

$$2G_n = R_{n+3}-R_{n+2}$$

we see that

$$A^n = \begin{pmatrix} G_n & G_{n+1} & G_{n-1} \\ G_{n-1} & G_n & G_{n-2} \\ G_{n-2} & G_{n-1} & G_{n-3} \end{pmatrix} = B_n.$$

We define

$$E_n = \begin{pmatrix} V_n & V_{n+1} & V_{n-1} \\ V_{n-1} & V_n & V_{n-2} \\ V_{n-2} & V_{n-1} & V_{n-3} \end{pmatrix}.$$

In this case, Teorem 7.1, Theorem 7.2 and Corollary 7.3 can be given as follows:

Theorem 7.4. *For all integer* $m,n \geq 0$, *we have*

(a) $B_n = A^n$

(b) $E_1 A^n = A^n E_1$

(c) $E_{n+m} = E_n B_m = B_m E_n$.

Proof.

(a) The proof is given in Theorem 7.1.

(b) Using (a) and definition of E_1, (b) follows.

(c) We have

$$\begin{aligned}
AE_{n-1} &= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n-1} & V_n & V_{n-2} \\ V_{n-2} & V_{n-1} & V_{n-3} \\ V_{n-3} & V_{n-2} & V_{n-4} \end{pmatrix} \\
&= \begin{pmatrix} 2V_{n-2}+V_{n-3} & 2V_{n-1}+V_{n-2} & 2V_{n-3}+V_{n-4} \\ V_{n-1} & V_n & V_{n-2} \\ V_{n-2} & V_{n-1} & V_{n-3} \end{pmatrix} \\
&= \begin{pmatrix} V_n & V_{n+1} & V_{n-1} \\ V_{n-1} & V_n & V_{n-2} \\ V_{n-2} & V_{n-1} & V_{n-3} \end{pmatrix} = E_n,
\end{aligned}$$

i.e. $E_n = AE_{n-1}$. From the last equation, using induction we obtain $E_n = A^{n-1}E_1$. Now

$$E_{n+m} = A^{n+m-1}E_1 = A^{n-1}A^m E_1 = A^{n-1}E_1 A^m = E_n B_m$$

and similarly,

$$E_{n+m} = B_m E_n.$$

Theorem 7.5. *For all integers $m, n,$ we have*

$$V_{n+m} = V_{n+1}G_{m-1} + V_n G_m + V_{n-1}G_{m-2}.$$

Proof. From the equation $E_{n+m} = E_n B_m = B_m E_n$ we see that an element of E_{n+m} is the product of row E_n and a column B_m. From the last equation we say that an element of E_{n+m} is the product of a row E_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices E_{n+m} and $E_n B_m$. This completes the proof.

Corollary 7.6. *For all integers $m, n,$ we have*

$$\begin{aligned} R_{n+m} &= R_{n+1}G_{m-1} + R_n G_m + R_{n-1}G_{m-2}, \\ C_{n+m} &= C_{n+1}G_{m-1} + C_n G_m + C_{n-1}G_{m-2}, \\ G_{n+m} &= G_{n+1}G_{m-1} + G_n G_m + G_{n-1}G_{m-2}, \\ B_{n+m} &= B_{n+1}G_{m-1} + B_n G_m + B_{n-1}G_{m-2}. \end{aligned}$$

8 CONCLUSIONS

The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. Sequences of integer number such as Fibonacci, Lucas, Pell, Jacobsthal are the most well-known second order recurrence sequences. For rich applications of these second order sequences in science and nature, one can see the citations in [28].

We introduce the generalized Pell-Padovan sequence (it's four special cases, namely, Pell-Padovan, Pell-Perrin, third order Fibonacci-Pell and third order Lucas-Pell sequences) and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences. For the application of Pell-Padovan numbers to quaternions and groups see [20] and [19], respectively. Third order sequences have many other applications. We now present one of them. The ratio of two consecutive Padovan numbers converges to the plastic ratio, α_P (which is given in (8.1) below), which have many applications to such as architecture, see [29]. Padovan numbers is defined by the third-order recurrence relations

$$P_{n+3} = P_{n+1} + P_n, \quad P_0 = 1, P_1 = 1, P_2 = 1.$$

The characteristic equation associated with Padovan sequence is $x^3 - x - 1 = 0$ with roots α_P, β_P and γ_P in which

$$\alpha_P = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \simeq 1.32471795724 \tag{8.1}$$

is called plastic number (or plastic ratio or plastic constant or silver number) and

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \alpha_P.$$

The plastic number is used in art and architecture. Richard Padovan studied on plastic number in Architecture and Mathematics in [30, 31].

As future work, we plan to study on the other third order and higher order generalized sequences.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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