British Journal of Mathematics & Computer Science

22(3): 1-10, 2017; Article no.BJMCS.33414

ISSN: 2231-0851

Viscosity Approximation Methods in Reflexive Banach Spaces with a Sequence of Contractions

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Authors' contributions

This work was carried out collaboratively between both authors. They both wrote the first draft of the manuscript and managed literature searches. The final manuscript was approved by both authors.

Article Information

DOI: 10.9734/BJMCS/2017/33414
Editor(s):
(1) Dariusz Jacek Jakobczak, Chair of Computer Science and Management in this Department, Technical University of Koszalin, Poland.
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(3) Ali Mutlu, Matmatics Manisa Celal Bayar University, Turkey.
Complete Peer review History: http://www.sciencedomain.org/review-history/19251

Received: 13th April 2017
Accepted: 17th May 2017
Published: 30th May 2017

Original Research Article

Abstract

The aim of this paper is to study viscosity approximation methods in reflexive Banach spaces. Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping $j\colon E \to E^*$, C a nonempty closed convex subset of E, μ_n, $n \geq 1$ a sequence of contractions on C and T_n, $n = 1, 2, 3, \cdots N$ a finite family of nonexpansive mappings on C. We show that under appropriate conditions on κ the implicit iterative sequence τ_κ defined by

$$\tau_\kappa = \kappa\mu_n(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa$$

where $\kappa \in (0,1)$ converges strongly to a common fixed point $\tau \in \bigcap_{n=1}^{k} F_{T_n}$. We further show that the results hold for an infinite family T_n, $n \in N$ of nonexpansive mappings.

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Keywords: *Viscosity approximation methods; reflexive Banach spaces; nonexpansive mappings; weakly sequentially continuous duality mapping; sequence of contractions; common fixed points; commuting nonexpansive mappings; implicit iteration.*

2010 Mathematics Subject Classification: 47H09, 47H10.

1 Introduction

Let E denote a reflexive Banach space which admits a weakly sequentially continuous duality mapping $j\colon E \to E^*$, and let E^* be the first dual of E. Let J be the normalized duality mapping $J : E \to 2^{E^*}$ defined by $Jx = \{\mu \in E^* : \langle x, \mu\rangle = \|x\|\|\mu\|; \|x\| = \|\mu\|\}$, $\forall\, x \in E$. The single-valued duality mapping will be denoted by j; and F_T will denote the set of fixed points of T given by $F_T = \{x \in E : Tx = x\}$.

We denote the strong convergence of the sequence $\{x_n\}$ in E to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Let E denote a real Banach space, C a closed convex nonempty subset of E. Given a mapping $T : C \longrightarrow C$, T is called ζ-Lipschitzian if there exists a constant $\zeta > 0$ such that

$$\| Tx - Ty \| \leq \zeta \| x - y \|, \quad \forall \quad pair \;\; x, y \in E.$$

T is nonexpansive if for every pair $x, y \in E$, $\| Tx - Ty \| \leq \| x - y \|$. T is said to be a contraction on C if a constant $\alpha \in (0, 1)$ exists such that for every pair $x, y \in E$, $\| Tx - Ty \| \leq \alpha \| x - y \|$. We denote by $\prod_C$ the set of all contractions on C.

Recently, K. Aoyama and Y. Kimura [1] introduced an iterative scheme by the viscosity approximation method with a sequence of contractions and proved the strong convergence of

$$y_{n+1} = \lambda_n f_n(y_n) + (1 - \lambda_n)T_n y_n, \quad y_1 \in C,$$

in a Hilbert space H, where C is a nonempty closed convex subset of H, $f \in \prod_C$ and $\lambda_n \in [0, 1]$.

In [2], H. K Xu proposed the iterative scheme $\{x_\kappa\}$ given by

$$x_\kappa = \kappa f(x_\kappa) + (1 - \kappa)Tx_\kappa$$

where $f \in \prod_C$, $\kappa \in$ (0,1) and T is a nonexpansive mapping on C, and proved the strong convergence of x_κ to a fixed point of T.

Again in [3], Yisheng and Rudong proved that as $\kappa \to 0$, $x_\kappa = P(\kappa f(x_\kappa) + (1 - \kappa)Tx_\kappa)$ converges to a fixed point of T in a Banach space.

The schemes above are major results which have emerged from the pioneering work of A. Moudafi [4] who introduced the explicit iterative scheme

$$x_{n+1} = \frac{1}{1 + \varepsilon_n} Tx_n + \frac{\varepsilon_n}{1 + \varepsilon_n} f(x_n), \quad x_1 \in C,$$

where ε_n is a sequence of positive numbers in R such that $\varepsilon_n \to 0$ as $n \to \infty$, f is a contraction on C and T is nonexpansive defined on C; and obtained results in Hilbert spaces. Such a method for approximating fixed points is called viscosity approximation method.

In this paper, motivated by the achievements of Aoyama & Kimura [1], Xu [2], Yisheng & Rudong [3] and Chang [5] we prove the strong convergence of the sequence

$$\tau_\kappa = \kappa \mu_n(\tau_\kappa) + (1 - \kappa)T_n \tau_\kappa$$

in a reflexive Banach space E which admits a weakly sequentially continuous duality mapping $j : E \to E^*$ to a common fixed point of the family T_n ,$n = 1, 2, \cdots, N$. This generalizes and improves several recent results. Particularly, it extends and improves Theorem 4.1 of [2] and Theorem 2.2 of [3].

2 Preliminaries

Lemma 2.1

(Xu, [2]) Let E be a Banach space and C a bounded closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping on C and $\mu \in \prod_C$. Given a real number $\kappa \in (0, 1)$, define a mapping $T_\kappa : C \to C$ by $T_\kappa x = \kappa\mu(x) + (1-\kappa)Tx \ \forall x \in C$ Then T_κ is a contraction on C.

Proof

Let $x, y \in C$. Then

$$\begin{aligned}
\|T_\kappa x - T_\kappa y\| &= \|\kappa\, \mu(x) + (1-\kappa)Tx - [\kappa\, \mu(y) + (1-\kappa)Ty]\| \\
&= \|\kappa\, [\mu(x) - \mu(y)] + (1-\kappa)[Tx - Ty]\| \\
&\leq \kappa\alpha\|x-y\| + (1-\kappa)\|x-y\| \\
&= (\kappa\alpha + (1-\kappa))\|x-y\| \\
&= (1-\kappa(1-\alpha))\|x-y\|
\end{aligned}$$

Since $1-\kappa(1-\alpha) \in (0, 1)$, T_κ is a contraction. ■

Theorem 2.2

Let E be a reflexive Banach space with dual E^* and bidual E^{**}. Let C be a bounded closed convex subset of E. Let μ_n be a sequence of contractions on C such that

$$\mu_n \in C^* \quad \forall\, n \geq 1$$

and

$$\mu_n(x) \leq \mu_{n+1}(x), \ \ \forall\, n, \ \ x \in E$$

. If μ_n converges pointwise on C to a contraction μ then the convergence is uniform.

Proof

Let $f_n(x) = \mu(x) - \mu_n(x)$ for each $n \in N$. Then f_n is a sequence of contractions on the compact set C such that $f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in C$and $n \in N$.
Moreover,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \{\mu(x) - \mu_n(x)\} = 0$$

Let $M_n = \sup\{f_n(x) : x \in C\}$ and let $\varepsilon > 0$ be given.
Also let

$$E_n = \{x \in C : f_n(x) < \varepsilon\} = f_n^{-1}((-\infty, \varepsilon))$$

Then E_n is open for each n and $E_n \subset E_{n+1}$ because $f_n(x) \geq f_{(n+1)}(x)$.
Since for each $x \in C$ $\lim\limits_{n\to\infty} f_n(x) = 0$ there exists $n \in N$ such that $f_n(x) < \varepsilon$ which implies $x \in E_n$. Thus $\bigcup_{n=1}^{\infty} E_n$ is an open cover for C and $\bigcup_{n=1}^{\infty} E_n = C$. Since C is compact there exists a finite subcover for C and in view of the fact that $E_n \subset E_{n+1}$, the largest of these also covers C. Hence there is $N \in N$ such that $E_N = C$ and this means that $f_N(x) < \varepsilon$ for all $x \in C$ and $n \geq N$. Thus $M_N \leq \varepsilon$ and since $M_n \geq 0$, $\lim\limits_{n\to\infty} M_n(x) = 0$. This indicates that the sequence f_n converges uniformly to 0 on C and therefore the sequence of contractions μ_n converges uniformly to μ on C.■

Lemma 2.3

(Chang, [5]) Let E be a Banach space and let C be a bounded closed convex subset of X. Let T_n, $n = 1, 2, \cdots, k$ be a finite family of commuting nonexpansive mappings defined on C into itself. Then $\bigcap_{n=1}^{k} F_{T_n} = F_T$, where $T = T_1T_2T_3 \cdots T_{\kappa-1}T_\kappa$

Proof

Let $x \in \bigcap_{n=1}^{k} F_{T_n}$. Then $x \in F_{T_n}$, for $n = 1, 2, \cdots, k$.
Hence

$$T_n x = x$$

for each $n = 1, 2, \cdots, k$. Thus $x \in F_T$ since $Tx = T_1T_2T_3 \cdots T_{k-1}T_k x = x$ and $\bigcap_{n=1}^{k} F_{T_n} \subset F_T$.
Next, suppose $x \in F_T$. Then

$$T_1T_2T_3 \cdots T_{k-1}T_k x = x.$$

Let $T = T_1$. Then $Tx = T_1x = x$ and $x \in F_{T_1}$
Again let $T = T_1T_2$. Then

$$x = Tx = T_1T_2x = T_2T_1x = T_2x$$

Hence $x \in F_{T_2}$.
Next, if $T = T_1T_2T_3$ then we have

$$x = Tx = T_1T_2T_3x = T_3T_2T_1x = T_3T_2x = T_3x$$

and this implies $x \in F_{T_3}$.
Finally, for $N \in N$ such that $N \geq 1$, let $T = T_1T_2T_3 \cdots T_{N-1}$ and assume that $x \in F_{T_n}$ for each $n = 1, 2, \cdots, k-1$. whenever $T_1T_2T_3 \cdots T_{N-1}x = x$
Now, if $T = T_1T_2T_3 \cdots T_{N-1}T_N$ then

$$x = Tx = T_1T_2T_3 \cdots T_{k-1}T_kx = T_kT_1T_2T_3 \cdots T_{N-1}x = T_Nx$$

Therefore $x \in F_{T_N}$. Thus, by induction if $T_1T_2T_3 \cdots T_{N-1}T_Nx = x$ then $T_nx = x$ for each $n = 1, 2, \cdots, N$ and so $x \in \bigcap_{k=1}^{N} F_{T_k}$.
This follows that

$$F_T \subset \bigcap_{n=1}^{k} F_{T_n}.$$

The conclusion is that $\bigcap_{k=1}^{N} F_{T_k} = F_T$. ■

Lemma 2.4

Let E be a Banach space and let C be a bounded closed convex subset of E. Let T_n, $n = 1, 2, \cdots, k$ be a finite family of commuting nonexpansive mappings defined on C into itself. Let $\bigcap_{n=1}^{k} F_{T_n} = F_T$, where $T = T_1T_2T_3 \cdots T_{k-1}T_k$. Then the mapping $T : C \to C$ is nonexpansive.

Proof

Let $x, y \in C$. Then $Tx = T_1T_2T_3 \cdots T_{k-1}T_k(x)$ and $Ty = T_1T_2T_3 \cdots T_{k-1}T_k(y)$
Now,

$$\|Tx - Ty\| = \|T_1T_2T_3 \cdots T_{-1}T_k(x) - T_1T_2T_3 \cdots T_{k-1}T_k(y)\|$$

Next we show that P_N is true $\forall\ N \in N$.

Let

$$P_N : \|T_1T_2T_3 \cdots T_{N-1}T_N(x) - T_1T_2T_3 \cdots T_{N-1}T_N(y)\| \leq \|x - y\| \ \ \forall \ \ x, y \in C$$

By hypothesis,

$$\|T_1x - T_1y\| \leq \|x - y\| \;\; \forall \;\; x, y \in C.$$

Therefore $P_N$is true when $N = 1$
Assuming that for some $k \in N$, P_k is true we get

$$\|T_1T_2T_3 \cdots T_{k-1}T_kx - T_1T_2T_3 \cdots T_{k-1}T_ky\| \leq \|x - y\| \;\; \forall \;\; x, y \in C.$$

Finally by commutation of the nonexpansive maps;

$$\|T_1T_2T_3 \cdots T_kT_{k+1}x - T_1T_2T_3 \cdots T_kT_{k+1}y\| \leq \|T_{k+1}x - T_{k+1}y\| \leq \|x - y\| \;\; \forall \;\; x, y \in C$$

Thus $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and T is nonexpansive. ■

Lemma 2.5

(Xu, [2]) Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers, $\{\beta_n\}$ a sequence in (0,1) and $\{\varphi_n\}$ a sequence in R such that $\alpha_{n+1} \leq (1-\beta_n)\alpha_n + \varphi_n$, $n \geq 0$, where $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\varphi_n| < \infty$. Then $\alpha_n \to 0$ as $n \to \infty$.

Lemma 2.6

(Yisheng, [3]) Let C be a nonempty closed convex subset of a reflexive Banach space E which satisfies Opial's condition. Suppose also that $T : C \to C$ is nonexpansive and $\{x_n\}$ is a sequence in C such that $x_n - Tx_n \to 0$. Then $x = Tx$.

Lemma 2.7

(Chang, [5]) Let E be a Banach space with dual E^*. Let $J : E \to 2^{E^*}$ defined by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \;\; \|x\| = \|f\|\}, \;\; \forall \;\; x \in E$ be the normalized duality mapping on E. Then $\forall \; x \in E$, $\forall \; j(x) \in J(x)$ and $\forall \; j(x+y) \in J(x+y)$, the following subdifferetial inequalities (i) and (ii) hold in E.

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$

(ii) $\|x\|^2 + 2\langle y, j(x)\rangle \leq \|x + y\|^2$

Lemma 2.8

(Xu, [2]) Let E be a real Banach space with dual E^* and let J be as in Lemma (2.7). If C is a closed convex subset of E and $\mu : C \to C$ is a contraction with coefficient $\alpha \in (0, 1)$, then

$$(1 - \alpha)\|x - y\|^2 \leq \langle (I - \mu)x - (I - \mu)y, j(x - y)\rangle \;\; \forall \;\; x, y \in C,$$

and $(I - \mu)$ is said to be strongly monotone where I is the identity operator.

3 Main Results

Theorem 3.1

Let E be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping $j : E \longrightarrow E^*$, C a bounded closed convex nonempty subset of E, μ_n, $n \in \mathbb{N}$, a sequence of contractions on C such that $\mu_n(\tau) \leq \mu_{n+1}(\tau)$, $\forall \; n, \; \tau \in C$, and T_n, $n = 1, 2, 3, \cdots, k$ a finite family of commuting nonexpansive mappings on C such that $\cap_{n=1}^{k} F_{T_n} \neq \emptyset$ and satisfies the condition

$$\cap_{n=1}^{k} F_{T_n} = F(T_1T_2T_3 \cdots T_{k-1}T_k) = F_T,$$

where $T = T_1T_2T_3\cdots T_{k-1}T_k$. Suppose also that for each $\mu_n \in \prod_C$, $\mu(p) \neq p$, $\forall\ p \in \cap_{n=1}^{N} F_{T_n}$, $\kappa \in (0,1)$ and the net $\{\tau_\kappa\}$ satisfies the following condition;

K_1 : $\| \tau_\kappa - T\tau_\kappa\| \longrightarrow 0\ \ as\ \ \kappa \longrightarrow 0$.

Then as $\kappa \longrightarrow 0$, the sequence

$$\tau_\kappa = \kappa\mu_n(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa \tag{3.1}$$

satisfies these two conditions;

K_2 : $\lim\limits_{\kappa\longrightarrow 0} \tau_\kappa = \tau^*$ exists;

K_3 : τ^* is the unique solution in $\cap_{n=1}^{N} F_{T_n}$ to the variational inequality

$$\langle (I-\mu)\tau^*, j(\tau-\tau^*)\rangle \geq 0,\ \ \forall\ \ \tau \in \cap_{n=1}^{N} F_{T_n}$$

Proof

From Theorem 2.2, $\mu_n \longrightarrow \mu$ uniformly on C. Thus (3.1) becomes

$$\tau_\kappa = \kappa\mu(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa.$$

Suppose $\lambda \in C$ is such that $\lambda \in \cap_{n=1}^{k} F_{T_n}$.
Then

$$\begin{aligned}\| \tau_\kappa - \{(1-\kappa)\lambda + \kappa\mu(\tau_\kappa)\} \| &= \| (\kappa\mu(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa) - \{(1-\kappa)\lambda + \kappa\mu(\tau_\kappa)\} \| \\ &= \| (1-\kappa)(T_n\tau_\kappa - \lambda) \| \\ &\leq (1-\kappa) \| \tau_\kappa - \lambda \| \end{aligned} \tag{3.2}$$

Again,

$$\| \tau_\kappa - \{(1-\kappa)\lambda + \kappa\mu(\tau_\kappa)\} \|^2 = \| (1-\kappa)(\tau_\kappa - \lambda) + \kappa(\tau_\kappa - \mu(\tau_\kappa)) \|^2 \tag{3.3}$$

By (ii) of Lemma 2.7,

$$(1-\kappa)^2 \| \tau_\kappa - \lambda \|^2 + 2\kappa(1-\kappa)\langle \tau_\kappa - \mu(\tau_\kappa),\ j(\tau_\kappa - \lambda)\rangle \leq \| (1-\kappa)(\tau_\kappa - \lambda) + \kappa(\tau_\kappa - \mu(\tau_\kappa)) \|^2$$

Therefore

$$(1-\kappa)^2 \| \tau_\kappa - \lambda \|^2 + 2\kappa(1-\kappa)\langle \tau_\kappa - \mu(\tau_\kappa),\ j(\tau_\kappa - \lambda)\rangle \leq \| \tau_\kappa - \{(1-\kappa)\lambda + \kappa\mu(\tau_\kappa)\} \|^2$$

$$\Longrightarrow\ 2\kappa(1-\kappa)\langle \tau_\kappa - \mu(\tau_\kappa), j(\tau_\kappa - \lambda)\rangle \leq \| \tau_\kappa - \{(1-\kappa)\lambda + \kappa\mu(\tau_\kappa)\} \|^2 - (1-\kappa)^2 \| \tau_\kappa - \lambda \|^2$$

$$\Longrightarrow\ 2\kappa(1-\kappa)\langle \tau_\kappa - \mu(\tau_\kappa), j(\tau_\kappa - \lambda)\rangle \leq 0$$

Which follows that,

$$\langle \tau_\kappa - \mu(\tau_\kappa), j(\tau_\kappa - \lambda)\rangle \leq 0. \tag{3.4}$$

To show that $\{\tau_\kappa : \kappa \in (0,1)\}$ is bounded we choose $\alpha \in (0,1)$ and use the fact that

$$\langle \mu(\tau_\kappa) - \mu(\lambda),\ j(\tau_\kappa - \lambda)\rangle \leq \alpha \| \tau_\kappa - \lambda \| \| \tau_\kappa - \lambda \| = \alpha \| \tau_\kappa - \lambda \|^2$$

to obtain

$$\begin{aligned}\langle \tau_\kappa - \mu(\tau_\kappa),\ j(\tau_\kappa - \lambda)\rangle &= \langle [\tau_\kappa - \lambda + \lambda - \mu(\lambda) + \mu(\lambda) - \mu(\tau_\kappa)],\ j(\tau_\kappa - \lambda)\rangle \\ &\leq \langle \tau_\kappa - \lambda, j(\tau_\kappa - \lambda)\rangle + \langle \lambda - \mu(\lambda),\ j(\tau_\kappa - \lambda)\rangle + \langle \mu(\lambda) - \mu(\tau_\kappa), j(\tau_\kappa - \lambda)\rangle \\ &\leq \| \tau_\kappa - \lambda \|^2 + \langle \lambda - \mu(\lambda),\ j(\tau_\kappa - \lambda)\rangle - \langle \mu(\tau_\kappa) - \mu(\lambda),\ j(\tau_\kappa - \lambda)\rangle \\ &\leq \| \tau_\kappa - \lambda \|^2 + \langle \lambda - \mu(\lambda),\ j(\tau_\kappa - \lambda)\rangle - \alpha \| \tau_\kappa - \lambda \|^2 \end{aligned}$$

$$\leq (1-\alpha) \parallel \tau_\kappa - \lambda \parallel^2 + \langle \lambda - \mu(\lambda),\ j(\tau_\kappa - \lambda)\rangle$$

Using (3.4), we get

$$(1-\alpha) \parallel \tau_\kappa - \lambda \parallel^2 + \langle \lambda - \mu(\lambda),\ j(\tau_\kappa - \lambda)\rangle \leq 0$$

Therefore,

$$(1-\alpha) \parallel \tau_\kappa - \lambda \parallel^2 \leq \langle \mu(\lambda) - \lambda,\ j(\tau_\kappa - \lambda)\rangle \leq \parallel \mu(\lambda) - \lambda \parallel \parallel \tau_\kappa - \lambda \parallel \tag{3.5}$$

From (3.5),
$\parallel \tau_\kappa - \lambda \parallel \leq \frac{1}{(1-\alpha)} \parallel \mu(\lambda) - \lambda \parallel$ which shows that $\{\tau_\kappa : \kappa \in (0,1)\}$ is bounded.
Thus the sets $\{T\tau_\kappa : \kappa \in (0,1)\}$ and $\{\mu(\tau_\kappa) : \kappa \in (0,1)\}$ are also bounded.
Next we show that $\lim_{\kappa \longrightarrow 0} \parallel \tau_\kappa - T_n\tau_\kappa \parallel = 0$ for each n.

$$\begin{aligned}\parallel \tau_\kappa - T_n\tau_\kappa \parallel &= \parallel \kappa\mu(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa - T_n\tau_\kappa \parallel \\ &= \parallel \kappa\mu(\tau_\kappa) - \kappa T_n\tau_\kappa \parallel \\ &= \kappa \parallel \mu(\tau_\kappa) - T_n\tau_\kappa \parallel \longrightarrow 0.\ as\ \kappa \longrightarrow 0\end{aligned}$$

Thus condition K_1 is satisfied.

Now suppose that $\{\tau_{\kappa_n}\}$ is a subsequence of $\{\tau_\kappa\}$. Since $\{\tau_\kappa\}$ is bounded and E is reflexive $\tau_{\kappa_n} \rightharpoonup \tau^*$ as $\kappa_n \longrightarrow 0$, where $\{\kappa_n\}$ is a sequence in $(0,1)$ such that $\kappa_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then by Lemma (2.6) and condition K_1 , $\tau^* \in \cap_{n=1}^k F_{T_n}$ Now, using (3.5), we obtain

$$\parallel \tau_{\kappa_n} - \tau^* \parallel^2 \leq \frac{1}{(1-\alpha)} \langle \mu(\tau^*) - \tau^*, j(\tau_{\kappa_n} - \tau^*)\rangle.$$

By the fact that j is weakly sequentially continuous,

$$j(\tau_{\kappa_n} - \tau^*) \longrightarrow j(\tau^* - \tau^*) = j(0) = 0\ as\ \kappa_n \longrightarrow 0.$$

Hence $\tau_{\kappa_n} \longrightarrow \tau^*$ as $\kappa_n \longrightarrow 0$
Next, we show that the entire net $\{\tau_\kappa\}$ converges to τ^*. To this end we choose another subsequence $\{\tau_{\gamma_n}\}$ of $\{\tau_\kappa\}$ such that $\gamma_n \longrightarrow 0$ as $n \longrightarrow \infty$.
Then by a similar argument as above

$$\tau_{\gamma_n} \longrightarrow \tau^{**} \in \cap_{n=1}^N F_{T_n}\ as\ \gamma_n \longrightarrow 0.$$

Finally, we show uniqueness of τ^*.
Since τ_κ is bounded, the sets $\{\tau_\kappa - \pi\}$ for any $\pi \in \cap_{n=1}^k F_{T_n}$ and $\{\tau_\kappa - \mu(\tau_\kappa)\}$ are also bounded and the fact that j is single-valued and weakly sequentially continuous from E to E^*, we get

$$\parallel \langle (I-\mu)\tau_{\kappa_n} - (I-\mu)\tau^*\rangle \parallel \longrightarrow 0\ as\ \tau_{\kappa_n} \longrightarrow \tau^*.$$

Now, letting $\psi = \parallel \langle \tau_{\kappa_n} - \mu(\tau_{\kappa_n}),\ j(\tau_{\kappa_n} - \pi)\rangle - \langle (I-\mu)\tau^*, j(\tau^* - \pi)\rangle \parallel$ we get

$\psi \leq \parallel \langle \tau_{\kappa_n} - \mu(\tau_{\kappa_n}), j(\tau_{\kappa_n} - \pi)\rangle - \langle (I-\mu)\tau^*, j(\tau_{\kappa_n} - \pi)\rangle \parallel + \parallel \langle (I-\mu)\tau^*,\ j(\tau_{\kappa_n} - \pi)\rangle - \langle (I-\mu)\tau^*,\ j(\tau^* - \pi)\rangle \parallel$

$\leq \parallel \langle (I-\mu)\tau_{\kappa_n} - (I-\mu)\tau^*,\ j(\tau_{\kappa_n} - \pi)\rangle \parallel + \parallel \langle (I-\mu)\tau^*,\ j(\tau_{\kappa_n} - \pi) - j(\tau^* - \pi)\rangle \parallel$

$\leq \parallel (I-\mu)\tau_{\kappa_n} - (I-\mu)\tau^* \parallel \parallel \tau_{\kappa_n} - \pi \parallel + \parallel \langle (I-\mu)\tau^*,\ j(\tau_{\kappa_n} - \pi) - j(\tau^* - \pi)\rangle \parallel \longrightarrow 0\ as\ \tau_{\kappa_n} \longrightarrow \tau^*$ and as $\kappa_n \longrightarrow 0$
Thus by (3.4),

$$\langle (I-\mu)\tau^*, j(\tau^* - \pi)\rangle = \lim_{\kappa_n \to 0} \langle I-\mu)\tau_{\kappa_n}, j(\tau_{\kappa_n} - \pi)\rangle \leq 0 \tag{3.6}$$

Similarly for any $\pi \in \cap_{n=1}^k F_{T_n}$, and by $\tau_{\gamma_n} \to \tau^{**}$ as $\gamma_n \to 0$, we get

$$\langle (I-\mu)\tau^{**}, j(\tau^{**} - \pi)\rangle = \lim_{\kappa_n \to 0} \langle (I-\mu)\tau_{\gamma_n}, j\tau_{\gamma_n} - \pi\rangle \leq 0 \tag{3.7}$$

Letting $\pi = \tau^{**}$ in (3.6) and $\pi = \tau^*$ in (3.7) we obtain

$$\langle (I-\mu)\tau^*, j(\tau^* - \tau^{**}\rangle \leq 0 \tag{3.8}$$

$$\langle (I-\mu)\tau^{**}, j(\tau^{**} - \tau^*)\rangle \leq 0 \tag{3.9}$$

Adding up (3.8) and (3.9) yields

$$\langle (I-\mu)\tau^*, j(\tau^* - \tau^{**})\rangle + \langle (I-\mu)\tau^{**}, j(\tau^{**} - \tau^*)\rangle \leq 0$$

which implies

$$\langle (I-\mu\tau^*, j(\tau^* - \tau^{**})\rangle - \langle (I-\mu)\tau^{**}, j(\tau^* - \tau^{**})\rangle \leq 0 \tag{3.10}$$

From (3.10) we get

$$\langle (I-\mu)\tau^* - (I-\mu)\tau^{**}, j(\tau^* - \tau^{**})\rangle \leq 0. \tag{3.11}$$

Combining (3.11) and Lemma 2.8 gives

$$(1-\alpha) \parallel \tau^* - \tau^{**} \parallel^2 \leq \langle (I-\mu)\tau^* - (I-\mu)\tau^{**}, j(\tau^* - \tau^{**})\rangle \leq 0.$$

Thus we have $\tau^* = \tau^{**}$ ■

Finally we show that $\tau^* \in \bigcap_{n=1}^k F(T_n)$ solves the variational inequality

$$\langle (I-\mu)\tau^*, j(\tau - \tau^*)\rangle \geq 0.$$

Since for each n, τ_κ solves the fixed point equation

$$\tau_\kappa = \kappa\mu_n(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa \;\; as \;\; \kappa \to 0,$$

we get
$\tau_\kappa - \mu(\tau_\kappa) = \kappa\mu(\tau_\kappa) - \mu(\tau_\kappa) + (1-\kappa)T\tau_\kappa$
which implies

$$\begin{aligned}
(I-\mu)\tau_\kappa &= \kappa\mu(\tau_\kappa) - \kappa T\tau_\kappa + T\tau_\kappa - \mu(\tau_\kappa) \\
&= \kappa\{\mu(\tau_\kappa) - T\tau_\kappa\} - \{\mu(\tau_\kappa) - T\tau_\kappa\} \\
&= (\kappa - 1)\{\mu(\tau_\kappa) - T\tau_\kappa\} \\
&= (1-\kappa)\{T\tau_\kappa - \mu(\tau_\kappa)\} \\
(I-\mu)\tau_\kappa &= (1-\kappa)\{T\tau_\kappa - \frac{1}{\kappa}[\tau_\kappa - (1-\kappa)T\tau_\kappa]\} \\
\kappa(I-\mu)\tau_\kappa &= (1-\kappa)(\kappa T\tau_\kappa - \tau_\kappa + T\tau_\kappa - \kappa T\tau_\kappa) \\
\kappa(I-\tau)\tau_\kappa &= (1-\kappa)(T\tau_\kappa - \tau_\kappa) \\
\kappa(I-\mu)\tau_\kappa &= -(1-\kappa)(I-T)\tau_\kappa \\
(I-\tau)\tau_\kappa &= -\frac{(1-\kappa)}{\kappa}(I-T)\tau_\kappa
\end{aligned}$$

Therefore for any $\tau \in \bigcap_{n=1}^k F(T_n)$,

$$\begin{aligned}
\langle (I-\mu)\tau_\kappa, \tau - \tau_\kappa\rangle &= -\frac{(1-\kappa)}{\kappa}\langle (I-T)\tau_(\kappa), \tau - \tau_\kappa\rangle \\
&= -\frac{(1-\kappa)}{\kappa}\langle (I-T)\tau_\kappa - (I-T)\tau, \tau - \tau_\kappa\rangle
\end{aligned}$$

$$= \frac{(1-\kappa)}{\kappa}\langle (I-T)\tau_\kappa - (I-T)\tau, \tau_\kappa - \tau\rangle$$

Since $(I-T)$ is monotone, $\frac{(1-\kappa)}{\kappa}\langle (I-T)\tau_\kappa - (I-T)\tau,\ \tau_\kappa - \tau\rangle \geq 0.$
Thus

$$\langle (I-\mu)\tau_\kappa,\ \tau - \tau_\kappa\rangle \geq 0.$$

Now, $\tau_\kappa \to \tau^*$ as $\kappa \to 0$; hence by virtue of the duality pairing between x and $j(x)$ for any $x \in E$ and any $j(x) \in J(x)$,

$$\langle (I-\mu)\tau^*,\ j(\tau - \tau^*)\rangle \leq 0$$

■

We next consider an infinite family $T_1, T_2, T_3, \cdots$ of nonexpansive commuting maps on C. For the purpose of achieving our objective, we group these mappings into equivalent classes each of size N where $N \in N$. Thus the iteration of study becomes

$$\tau_k = \kappa\mu_n(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa, \ \ \tau_1 \in C, \quad n \geq 1$$

where $T_n = T_n(\text{mod N})$.

Lemma 3.2

Let E be a nonempty set. Then all equivalent classes of E are disjoint and E is the union of its equivalent classes.

Theorem 3.3

Let E be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping $j : E \to E^*$, C a bounded closed convex nonempty subset of E, $\mu_n, n \in N$, a sequence of contractions on C such that $\mu_n\ (\tau) \leq \mu_{n+1}(\tau)$, $\forall\ n,\ \tau \in C$, and T_n, $n = 1, 2, 3, \cdots$, an infinite family of commuting nonexpansive mappings on C such that $\bigcap_{n=1}^{N} F_(T_n) \neq \emptyset$ and satisfies the condition $\bigcap_{n=1}^{N} F_{T_n} = F(T_1T_2T_3 \cdots T_{N-1}T_N) = F_T$, where $T = T_1T_2T_3 \cdots T_{N-1}T_N$. Suppose also that for each $\mu_n \in \prod_C$, $\mu(p) \neq p$, $\forall\ p \in \bigcap_{n=1}^{N} F_{T_n}$, $\kappa \in (0, 1)$ and the net $\{\tau_\kappa\}$ satisfies the following condition;

K_1: $\|\tau_\kappa - T\tau_\kappa\| \to 0$ as $\kappa \to 0$.
Then as $\kappa \to 0$, the sequence

$$\tau_\kappa = \kappa\mu_n(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa$$

where $T_n = T_n(\text{mod N})$ satisfies these two conditions;

K_2 : $\lim_{\kappa\to 0} \tau_\kappa = \tau^*$ exists;

K_3: τ^* is the unique solution in $\bigcap_{n=1}^{N} F_{T_n}$ to the variational inequality

$$\langle (I-\mu)\tau^*, j(\tau - \tau^*)\rangle \geq 0,\ \forall\ \tau \in \bigcap_{n=1}^{N} F_{T_n}$$

Proof

The proof is done by partitioning the infinite family T_n, $\ n = 1, 2, 3, \cdots$ into equivalence classes of size N, for a positive integer N. Thus, all the assumptions made for the finite number of nonexpansive maps in Theorem 3.1 hold for each class. Again, since equivalent classes are disjoint, there are no spillovers into other classes. Therefore any result that is true for one class will also hold for other classes. Consequently, the result is a priori true from Theorem 3.1. ■

4 Conclusion

In [2], Xu considered an implicit iteration;

$$x_\kappa = \kappa f(x_\kappa) + (1-\kappa)Tx_\kappa \tag{4.1}$$

and showed that as $\kappa \to 0$ $x_\kappa \to x^* \in F_T$. He chose a fixed $f \in \prod_C$ and proved in a general Banach space (Theorem 4.1) and specifically in a Hilbert space (Theorem 3.1) the strong convergence of x_κ .

Yisheng and Rudong in [3, Theorem 2.2] proposed an implicit iterative scheme for the sequence x_κ defined by

$$x_\kappa = P(\kappa f(x_\kappa) + (1-\kappa)Tx_\kappa) \tag{4.2}$$

and demonstrated in a reflexive Banach space the strong convergence of x_κ. Again they chose a fixed contraction f on C.

The iterative scheme

$$\tau_\kappa = \kappa\mu_n(\tau_\kappa) + (1-\kappa)T_n\tau_\kappa \tag{4.3}$$

with a sequence of contractions and a finite family of nonexpansive mappings is a more general sequence than (4.1) and (4.2). Thus Theorem 3.1 is a generalization of [2, Theorem 4.1] and [3, Theorem 2.2].

Competing Interests

Authors have declared that no competing interests exist.

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