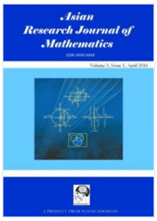

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A New Lifetime Model with a Bounded Support

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, we introduce a new probability model with a bounded support which possesses an increasing and bathtub failure rate functions. Numerous properties such as the moments, moment generating function, mean deviations, order statistics, moments of order statistics, joint and conditional distributions of order statistics are explored in explicit form. Statistical inferences by maximum likelihood method is considered, and we used simulation studies to access the proposed estimation procedure. An application of the proposed model to a real data set is presented for illustration.

Keywords: Hazard function; moments; maximum likelihood estimation.

2010 Mathematics Subject Classification: 62E05, 62F10, 62F12.

1 Introduction

In probability modeling, numerous probability distributions with a bounded support were introduced and studied in various literature, for instance, the continuous uniform (U) distribution which has

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support on a closed and bounded interval $[a, b]$, Beta (B) on $[0, 1]$, U-quadratic (Uq) on $[a, b]$, truncated normal (TN) on $[a, b]$, log normal (LN) on $[0, 1]$, Logit-normal (LtN) on (0,1), Irwin-Hall (IH) distribution on (0,1), minimax (0,1) by [1] and recently compounded with discret Weibull and inverse Weibull by [2, 3], triangular (Tr) distribution on [a, b], Kumaraswamy (Kw) on $(0, 1)$ by [4], exponentiated Kumaraswamy (EKw) on $(0, 1)$ by [5], transmuted Kumaraswamy (TKw) on (0,1) by [6], Topp-Leone (TL) on [0,1] by [7], Arcsine (Ac) on $[a, b]$, Mustapha type-I [a,b] by [8], among others.

Distributions defined on a unit interval play a vital role in the modeling of random phenomena. In recent years, several probability generators were proposed and studied which are more flexible in terms of failure rate and density than the parent distributions. For example (i) beta-G by [9] lead to so many distributions such as beta Weibull [10], beta Gumbel [11], beta exponentiated Weibull [12] and so on. (ii) Kumaraswamy-G leads to Kumaraswamy Weibull [13], Kumaraswamy Gumbel [14], Kumaraswamy Burr XII [15], etc. (iii) Topp-Leone Burr-XII by [16] due to Topp-Leone-G by [17], among many others. So, initially, we are highly motivated due to the fact that distributions with support on unit interval play an important role in probability modeling. Also, our aim is to propose a new lifetime model with the support on the unit interval, to explore some of its important properties and provide its application. The new distribution is derived from the general form of exponential functions. We hoped that the new distribution will lead to some flexible-G family that are alternative and better than many existing distributions.

In section 2, we present the new proposed distribution and provide several mathematical and statistical properties of the new model. In section 3, parameter estimation by maximum likelihood method is discussed. In section 4, real data application is provided. Conclusions in section 5.

2 New Model and Properties

In this section, we start by presenting the cumulative distribution function of the new probability model with parameter $\alpha > 0$ and $x \in [0, 1]$ as

$$F(x) = e^{x^{\alpha} \ln 2} - 1. \tag{2.1}$$

where the corresponding density, survival, hazard and reverse hazard functions are respectively given by

$$f(x) = \alpha \ln 2\, x^{\alpha-1} e^{x^{\alpha} \ln 2} \qquad 0 \leq x \leq 1, \tag{2.2}$$

$$h(x) = \frac{\alpha \ln 2\, x^{\alpha-1} e^{x^{\alpha} \ln 2}}{2 - e^{x^{\alpha} \ln 2}} \qquad 0 \leq x < 1. \tag{2.3}$$

The probability density function given by (2.2) can be presented in a series of the form

$$f(x) = \alpha \sum_{i=0}^{\infty} (i!)^{-1} (\ln 2)^{i+1} x^{\alpha(i+1)-1}\,. \tag{2.4}$$

The limiting behavior of the density of the new model as $x \to 0$ is (i) $\ln 2$ when $\alpha = 1$, (ii) 0 when $\alpha > 1$, (iii) ∞ when $\alpha < 1$ and (iv) as $x \to 1$ and for all $\alpha > 0$ the density goes to $\alpha \ln 4$. For the hazard rate function, (i) as $x \to 0$ the limiting behavior of h(x) is $\ln 2$ when $\alpha = 1$, (ii) 0 when $\alpha > 1$, (iii) ∞ when $\alpha < 1$ and (iv) as $x \to 1$ and for all $\alpha > 0$, $h(x) \to \infty$.

Theorem 2.1. *The probability density function f(x) given by (2.2) is monotone increasing function for $\alpha \geq 1$ and bathtub for $\alpha < 1$.*

Proof. We consider the $\log f(x) = \log(\alpha \ln 2) + (\alpha - 1)\log x + x^{\alpha} \ln 2$, and that, $(\log f(x))' = \frac{(\alpha-1)+\alpha x^{\alpha} \ln 2}{x} > 0$ for $\alpha \geq 1$, hence increasing function.

For $\alpha < 1$ let $\eta(x) = (\alpha - 1) + \alpha x^{\alpha} \ln 2$, then, the root of $\eta(x) = 0$, say $x_0 = \left(\frac{1-\alpha}{\alpha \ln 2}\right)^{\frac{1}{\alpha}}$, thus, $(\log f(x))' < 0$ for $0 < x < x_0$, $(\log f(x_0))' = 0$ and $(\log f(x))' > 0$ for $x_0 < x < 1$, hence f(x) is bathtub shaped. □

Theorem 2.2. *The hazard rate function $h(x)$ given by (2.3) is increasing function for $\alpha \geq 1$.*

Proof. According to [18], we get $(\log h(x))' = \frac{(\alpha-1)}{x} + \alpha x^{\alpha-1} \ln 2 + \frac{\alpha x^{\alpha-1} e^{x^{\alpha} \ln 2} \ln 2}{2 - e^{x^{\alpha} \ln 2}}$, thus, for $\alpha \geq 1$, $(\log h(x))' > 0$, hence, $h(x)$ is increasing function. □

It is also shown in Fig. 1, (ii), that, $h(x)$ can take bathtub shaped for $\alpha < 1$. Figure 1 shows the plots of density $(f(x))$ (i) and hazard function $(h(x))$ (ii) of the new distribution for some values of parameter α.

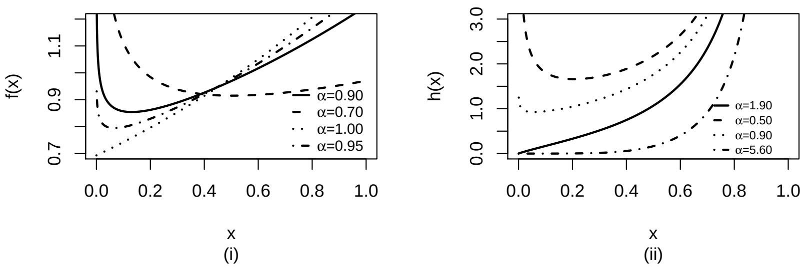

Fig. 1. Plots of the pdf (i) and hrf (ii) for some selected values of α

2.1 Quantile and moments

The quantile function $\zeta(.)$ of the new probability model can be used for generating random data distributed according to (2.2) through generating data from uniform distribution. The quantile function is given by

$$\zeta(p) = \left(\frac{\ln(p+1)}{\ln 2}\right)^{\frac{1}{\alpha}}. \tag{2.5}$$

The Median (M) of the proposed distributions can be obtained directly by substituting $p = 1/2$ as

$$M = \left(\frac{\ln(1.5)}{\ln 2}\right)^{\frac{1}{\alpha}}. \tag{2.6}$$

Fig. 3 shows that, the Median is increasing function in α.

Now, we compute the r^{th} moment and moment generating function of the new model which can be used directly to study some features and characteristics of the new distribution, such as the mean, variance, skewness, and kurtosis etc.

Theorem 2.3. *For a random variable X with pdf given by (2.2) and for $\alpha > 0$, $r \in \mathbb{N}$, then*

$$E[X^r] = \int_0^1 \alpha \ln 2\, x^{\alpha+r-1} e^{x^{\alpha} \ln 2} dx = \alpha \sum_{i=0}^{\infty} \frac{(\ln 2)^{i+1}}{i!(\alpha(i+1)+r)}. \tag{2.7}$$

Proposition 2.4. *If X is a random variable with (2.2) and for $\alpha, r \in \mathbb{N}$, $\gamma = \frac{\alpha+r-1}{\alpha} \in \mathbb{N}$, then,*

$$E[X^r] = 2\sum_{i=0}^{\gamma-1} \frac{(\gamma-1)!\,(-1)^{i-\gamma+1}}{(\ln 2)^{\gamma-i-1}\, i!}. \tag{2.8}$$

Proof. See subsection 2.32, under series expantion 2.33, equation 5^* in [19, p.108]. □

Corollary 2.5. *If X is a random variable that follow (2.2), then, for α, $r \in \mathbb{N}$,*

$$E[X^r] = \tfrac{2(\ln 2)^2 - 4\ln 2 + 2}{(\ln 2)^2},\ for\ \tfrac{\alpha+r-1}{\alpha} = 3,$$

$$E[X^r] = \tfrac{2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6}{(\ln 2)^3},\ for\ \tfrac{\alpha+r-1}{\alpha} = 4.$$

Proof. See subsection 2.32, under series expantion 2.33, equation 8^* and 9^* respectively in [19]. □

Fig. 2 provide the plots of the mean (μ) and variance (σ^2) of the new model, showing that the mean is increases in $\alpha > 0$ and the variance is unimodal as $\alpha > 0$ increases.

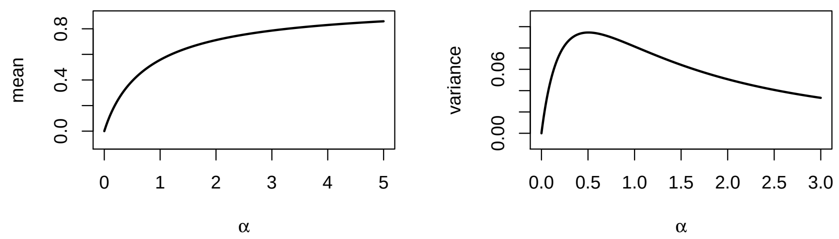

Fig. 2. Plots of the mean and variance for $\alpha > 0$

The moment generating function (*mgf*) of X can be obtain by putting (2.7) in the expansion of $M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$ as

$$M_X(t) = \alpha \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^r\,(\ln 2)^{i+1}}{i!\, r!\, (\alpha(i+1)+r)}. \tag{2.9}$$

Furthermore, the influence of the parameter α on the skewness and kurtosis of X can also be analyzed using the Bowley skewness (B) and Moores kurtosis (M), which are given by

$$B = \frac{\zeta(3/4) + \zeta(1/4) - 2\,\zeta(2/4)}{\zeta(3/4) - \zeta(1/4)} \quad \text{and} \quad M = \frac{\zeta(3/8) - \zeta(1/8) + \zeta(7/8) - \zeta(5/8)}{\zeta(6/8) - \zeta(2/8)}$$

respectively, where $\zeta(.)$ is given by (2.5). Fig. 3 demonstrated that, the skewness of the new distribution decreases as α increases while the kurtosis is decreasing then increasing (bathtub) as α increases.

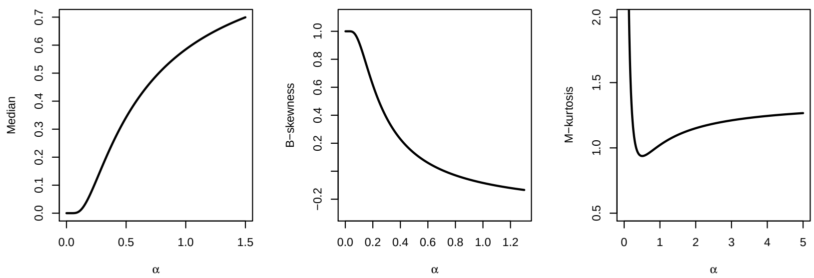

Fig. 3. Plots of the median, B-skewness and M-kurtosis of the new model for $\alpha > 0$

2.2 Mean deviations

The mean deviations of a random variable X about the mean is defined by $\delta_1(X) = E(|X - \mu_1|)$ and the mean deviations about the median (M) is $\delta_2(X) = E(|X - M|)$ which can be expressed as

$$\delta_1(X) = 2\mu_1 F(\mu_1) - 2m_{1(\mu_1)} \quad \text{and} \quad \delta_2(X) = \mu_1 - 2m_{1(M)}$$

respectively, where $\mu_1 = E(X)$, $F(\mu_1)$ can be computed from (2.1), M median of X which can be obtained from (2.6) and $m_{1(.)}$ is the first incomplete moment of X given as

$$m_1(t) = \int_0^t \alpha x^\alpha \ln 2 e^{x^\alpha \ln 2} dx = \alpha \sum_{i=0}^{\infty} \frac{(\ln 2)^{i+1} t^{\alpha(i+1)+1}}{i!(\alpha(i+1)+1)}, \tag{2.10}$$

thus, δ_1 and δ_2 can be computed by setting $t = \mu_1$ and $t = M$ in (2.10) respectively.

2.3 Order statistics

In this part, we obtain the density, r^{th}-moment, joint density and conditional density of the order statistics for a sample of independent observation obtained from the new distribution. The density of the j^{th} order statistic, $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, $j = 1, 2, 3, \cdots, n$, obtained from a random sample of size n from the new distribution can be presented in a series of the form

$$f_{X_{j:n}}(x; \alpha) = \sum_{i=0}^{n-j} \frac{n!\,(-1)^i}{(j-1)!(n-j-i)!\,i!} f(x)(F(x))^{j+i-1}, \tag{2.11}$$

where $f(x)$ and $F(x)$ are given by (2.2) and (2.1) respectively, thus,

$$f_{X_{j:n}}(x; \alpha) = \sum_{i=0}^{n-j} \sum_{k=0}^{j+i-1} \begin{pmatrix} i+j-1 \\ k \end{pmatrix} \frac{\alpha \;\ln 2\, n!\,(-1)^{2i+j-k-1}\, x^{\alpha-1}}{(j-1)!(n-j-i)!\,i!} e^{(k+1)x^\alpha \ln 2}. \tag{2.12}$$

The r^{th} moment of the j^{th} order statistics is computed using (2.12) as

$$E(X_{j:n}^r) = \sum_{i=0}^{n-j} \sum_{k=0}^{j+i-1} \sum_{w=0}^{\infty} \begin{pmatrix} i+j-1 \\ k \end{pmatrix} \frac{\alpha\, n!\,(-1)^{2i+j-k-1}(k+1)^w\,(\ln 2)^{w+1}}{(j-1)!(n-j-i)!\,i!\,w!(\alpha(w+1)+r)}. \tag{2.13}$$

Theorem 2.6. *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, be independent observations from the new model with pdf and cdf given by (2.2) and (2.1) respectively, then, for $1 \leq r < s \leq n$ and $0 \leq x_r \leq x_s \leq 1$,* *we have the joint density of* $X_{r:n}$ *and* $X_{s:n}$ *as*

$$f_{(X_{r:n},X_{s:n})}(x_r, x_s; \alpha) = \sum_{i=0}^{n-s}\sum_{k=0}^{r-1}\sum_{l=0}^{s-r-1}\begin{pmatrix} n-s \\ i \end{pmatrix}\begin{pmatrix} r-1 \\ k \end{pmatrix}\begin{pmatrix} s-r-1 \\ k \end{pmatrix} \times \psi_{i,k,l,n,r,s}(\alpha)\, {x_r}^{\alpha-1}\, {x_s}^{\alpha-1}\, e^{(k+l+1){x_r}^{\alpha}\ln 2}\, e^{(l+i-s-r){x_s}^{\alpha}\ln 2}, \tag{2.14}$$

where $\psi_{i,k,l,n,r,s}(\alpha) = \frac{\alpha^2(\ln 2)^2\, n!\, (-1)^{r+l+i-k-1}\, 2^{n-s-i}}{(r-1)!(n-s)!(s-r-1)!}$.

Proof: Using $f_{(X_{r:n},X_{s:n})}(x_r, x_s; \alpha) = \frac{n!\, f(x_r) f(x_s) F^{r-1}(x_r)}{(r-1)!(n-s)!(s-r-1)!}(1-F(x_s))^{n-s}(F(x_s)-F(x_r))^{s-r-1}$.

Theorem 2.7. *Let* $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$*, be independent observations from the new model with pdf and cdf given by (2.2) and (2.1) respectively, then, for* $1 \le r < s \le n$ *and* $0 \le x_r \le x_s \le 1$*, the conditional distribution of* $X_{s:n}$ *given that* $X_{r:n} = x_r$ *is given by*

$$f_{X_{s:n}|X_{r:n}}(x_s|x_r; \alpha) = \sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}\sum_{k=0}^{\infty}\begin{pmatrix} s-r-1 \\ i \end{pmatrix}\begin{pmatrix} n-s \\ j \end{pmatrix}\begin{pmatrix} -n+r \\ k \end{pmatrix} \times (-1)^{i+j+k}\, \alpha \ln 2\, x_s^{\alpha-1}\, e^{x_s^{\alpha}\ln 2}\, (e^{x_s^{\alpha}\ln 2}-1)^{s-r-k-j-1}\, (e^{x_r^{\alpha}\ln 2}-1)^{i+k}. \tag{2.15}$$

Proof: By using $f_{X_{s:n}|X_{r:n}}(x_s|x_r; \alpha) = \frac{(n-r)!\, f(x_s)}{(n-s)!(s-r-1)!}\,\frac{[F(x_s)-F(x_r)]^{s-r-1}[1-F(x_s)]^{n-s}}{[1-F(x_r)]^{n-r}}$.

3 Parameter Estimation

In this section, we discussed the parameter estimation by the method of maximum likelihood estimation and a simulation study is performed to assess the performance of the maximum likelihood method.

3.1 Maximum likelihood estimation

In this section, we estimate the unknown parameters of the new probability distribution by the method of maximum likelihood; we also investigate the existence of the maximum likelihood estimates under some possible conditions.

Let, X_1, X_2, $\cdots$, X_n, be a random sample of size n obtained from the new distribution, the log-likelihood function $(\log \ell(\alpha))$, is given by

$$\log \ell(\alpha) = n \log \alpha + n \log(\ln 2) + (\alpha - 1)\sum_{i=0}^{n}\log x_i + \log 2\sum_{i=0}^{n} x_i^{\alpha}, \tag{3.1}$$

hence, we can determine the maximum likelihood estimate of α, say $\hat{\alpha}$ by the numerical solution of (3.2) when equated to zero using mathematical software such as `R and mathematica.`

$$\frac{\partial \log \ell(\alpha)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=0}^{n}\log x_i + \log 2\sum_{i=0}^{n} x_i^{\alpha}\, \log x_i. \tag{3.2}$$

For a very large sample of size, we can set up the asymptotic distribution of α based on the approximation to normal distribution as follows by applying *lemma* 3.1 in *theorem* 3.2.

Lemma 3.1. *Let the random variable* $X \sim$ *(2.2), then,* $E[X^{\alpha}(\log X)^2] = \frac{\partial^2}{\partial t^2}E[X^{\alpha+t}]|_{t=0}$.

Theorem 3.2. *The maximum likelihood estimator $\hat{\alpha}$ of α is consistent estimator and $\sqrt{n}(\hat{\alpha}-\alpha)$ is asymptotically normal with mean $\mathbf{0}$ and variance $\boldsymbol{I}^{-1}$, where $\boldsymbol{I} = -\frac{1}{n}E(\partial^2(\log \ell(\alpha))/\partial\alpha^2)$ and*

$$E(\partial^2(\log \ell(\alpha))/\partial\alpha^2) = -\frac{n}{\alpha^2} + \log 2 \sum_{i=0}^{n} \frac{\partial^2}{\partial t^2} E[X^{\alpha+t}]|_{t=0}. \tag{3.3}$$

The existence of maximum likelihood estimates of a probability models under some certain conditions was considered and studied by many authors in various literature. For example, [20] studied the existence and uniqueness of the maximum likelihood estimators of the exponential geometric distribution, [21] for exponential poisson distribution, [22] for the four parameter exponentiated BurrXII poisson, others includes [23, 24, 25, 26, 27, 28] and [29] for the exponentiated exponential binomial, generalized exponential-power series, exponential-logarithmic, exponential-power series, generalized Burrxii-poisson, generalized half logistic poisson and complementary exponentiated BurrXII Poisson distributions respectively.

The following analyzed the existence of the MLE of α under some sufficient conditions.

Theorem 3.3. *Let $\jmath(\alpha; x_i)$ be the function on the right hand side of (3.2), then, the root of $\jmath(\alpha; x_i) = 0$ lies in the interval $\left(\frac{-n}{(1+\ln 2)\sum_{i=1}^{n}\log x_i}, \frac{-n}{\sum_{i=1}^{n}\log x_i}\right)$.*

Proof. Let $g(\alpha; x_i) = \log 2 \sum_{i=1}^{n} x^{\alpha} \log x_i$, then,
$\lim_{\alpha \to 0} g(\alpha; x_i) = \log 2 \sum_{i=1}^{n} \log x_i$ and $\lim_{\alpha \to\infty} g(\alpha; x_i) = 0$, therefore,
$\jmath(\alpha; x_i) = \frac{n}{\alpha} + \sum_{i=0}^{n} \log x_i + g(\alpha; x_i) > \frac{n}{\alpha} + \sum_{i=0}^{n} \log x_i + \lim_{\alpha \to 0} g(\alpha; x_i) = \frac{n}{\alpha} + (1+\ln 2)\sum_{i=1}^{n} \log x_i$,
hence, $\jmath(\alpha; x_i) > 0$ if $\alpha > \frac{-n}{(1+\ln 2)\sum_{i=1}^{n}\log x_i}$.
On the other hand,
$\jmath(\alpha; x_i) = \frac{n}{\alpha} + \sum_{i=0}^{n} \log x_i + g(\alpha; x_i) < \frac{n}{\alpha} + \sum_{i=0}^{n} \log x_i + \lim_{\alpha \to\infty} g(\alpha; x_i) = \frac{n}{\alpha} + \sum_{i=1}^{n} \log x_i$, hence,
$\jmath(\alpha; x_i) < 0$ if $\alpha < \frac{-n}{\sum_{i=1}^{n}\log x_i}$.
Thus, the root of $\jmath(\alpha; x_i) = 0$ lies in the interval $\left(\frac{-n}{(1+\ln 2)\sum_{i=1}^{n}\log x_i}, \frac{-n}{\sum_{i=1}^{n}\log x_i}\right)$. □

3.2 Simulation study

In this subsection, we investigate the performance of the maximum likelihood estimates based on the simulation study; we generate ten thousand samples from the new model each of sample size n (n= 50, 100, 200, and 300). The estimated values are computed by the numerical solution of the nonlinear equation (3.2) using `nlminb in R-software`. The sample size (n), actual value (α), estimated value ($\hat{\alpha}$) and standard deviations ($sd(\hat{\alpha})$) for some selected parameter values are given below in Table 1. The result in Table 1 shows that the proposed methods performed consistently in both the small and large sample sizes, also increasing the sample size decreases the standard deviation of the MLEs.

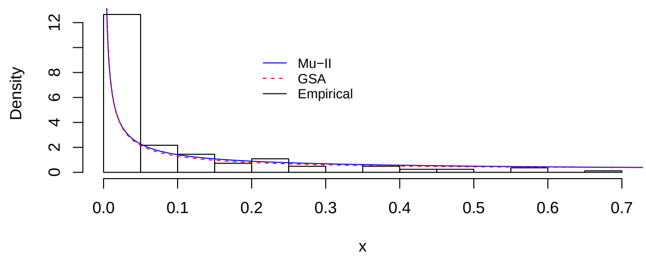

Fig. 4. Plot of histogram and the fitted distribution for the given dataset

Table 1. Estimated values and standard deviations for some selected values of parameter α for the Maximum likelihood estimates

α	n	$\hat{\alpha}$	$sd(\hat{\alpha})$	n	$\hat{\alpha}$	$sd(\hat{\alpha})$	n	$\hat{\alpha}$	$sd(\hat{\alpha})$	n	$\hat{\alpha}$	$sd(\hat{\alpha})$
0.5	50	0.5102	0.0809	100	0.5052	0.0565	200	0.5033	0.0388	300	0.5021	0.0315
0.1		0.1020	0.0162		0.1010	0.0113		0.1007	0.0078		0.1000	0.0063
0.9		0.9184	0.1457		0.9093	0.1018		0.9059	0.0698		0.9038	0.0567
1.9		1.9388	0.3076		1.9196	0.2148		1.9124	0.1475		1.9080	0.1196
1.5		1.5306	0.2428		1.5155	0.1696		1.5073	0.1155		1.5056	0.0945
2.0		2.0408	0.3238		2.0179	0.2194		2.0114	0.1542		2.0073	0.1254
0.6		0.6133	0.0959		0.6056	0.0656		0.6035	0.0465		0.6017	0.0379
0.2		0.2044	0.0320		0.2019	0.0219		0.2012	0.0155		0.2006	0.0127
3.5		3.5775	0.5594		3.5325	0.3828		3.5204	0.2714		3.5099	0.2215
1.8		1.8399	0.2877		1.8167	0.1969		1.8105	0.1396		1.8051	0.1139
2.5		2.5503	0.3955		2.5317	0.2737		2.5117	0.1920		2.5067	0.1561
1.0		1.0207	0.1591		1.0097	0.1107		1.0049	0.0770		1.0039	0.0636

4 Illustration

We fitted the new model to real data set and we compare the fit with the generalized standard arcsine distribution (GSA) with pdf given by $f(x) = \sin(\pi\alpha)\pi^{-1}x^{-\alpha}(1-x)^{\alpha-1}$, $\alpha > 0$ and $x \in (0, 1)$. We estimate the parameters of the models by the maximum likelihood, the Akaike information criteria (AIC) and Bayesian information criteria (BIC) are used to compare the fitted models. The data set is provided by [30] and recently studied by [6]. The data set are from a study on anxiety performed in a group of 166 normal women, that is, outside of a pathological clinical picture in Townsville, Queensland, Australia: 0.01, 0.17, 0.01, 0.05, 0.09, 0.41, 0.05, 0.01, 0.13, 0.01, 0.05, 0.17, 0.01, 0.09, 0.01, 0.05, 0.09, 0.09, 0.05, 0.01, 0.01, 0.01, 0.29, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.09, 0.37, 0.05, 0.01, 0.05, 0.29, 0.09, 0.01, 0.25, 0.01, 0.09, 0.01, 0.05, 0.21, 0.01, 0.01, 0.01, 0.13, 0.17, 0.37, 0.01, 0.01, 0.09, 0.57, 0.01, 0.01, 0.13, 0.05, 0.01, 0.01, 0.01, 0.01, 0.09, 0.13, 0.01, 0.01, 0.09, 0.09, 0.37, 0.01, 0.05, 0.01, 0.01, 0.13, 0.01, 0.57, 0.01, 0.01, 0.09, 0.01, 0.01, 0.01, 0.01, 0.01, 0.01, 0.05, 0.01, 0.01, 0.01, 0.13, 0.01, 0.25, 0.01, 0.01, 0.09, 0.13, 0.01, 0.01, 0.05, 0.13, 0.01, 0.09, 0.01, 0.05, 0.01, 0.05, 0.01, 0.09, 0.01, 0.37, 0.25, 0.05, 0.05, 0.25, 0.05, 0.05, 0.01, 0.05, 0.01, 0.01, 0.01, 0.17, 0.29, 0.57, 0.01, 0.05, 0.01, 0.09, 0.01, 0.09, 0.49, 0.45, 0.01, 0.01, 0.01, 0.05, 0.01, 0.17, 0.01, 0.13, 0.01, 0.21, 0.13, 0.01, 0.01, 0.17, 0.01, 0.01, 0.21, 0.13, 0.69, 0.25, 0.01, 0.01, 0.09, 0.13, 0.01, 0.05, 0.01, 0.01, 0.29, 0.25, 0.49, 0.01, 0.01. The result in table 2 shows that the new model (Mu-II) has the smallest value of the AIC, BIC and CAIC thus Mu-II fit the data better than the GSA distribution. Figure 4 shows the plot of the histogram and the fitted distribution of the given dataset.

Table 2. MLEs, $\ell(\alpha)$, AIC, BIC and CAIC for the given data set

Model	MLE	$\ell(\alpha)$	AIC	BIC	CAIC
Mu-II	$\hat{\alpha} = 0.2324$	177.84	-353.68	-350.57	-353.65
GSA	$\hat{\alpha} = 0.7542$	172.31	-342.63	-339.51	-342.60

5 Conclusions

We have proposed and studied a new lifetime distribution with an increasing and bathtub-shaped hazard rate functions named *Mustapha type-II distribution.* We provide several mathematical and statistical properties of the new distribution, such as explicit algebraic expressions for the r^{th} moments, moment generating function, mean deviations, order statistics, moments of order statistics, joint and conditional distributions of order statistics. The estimation of the model parameters was approached by the maximum likelihood estimate; we also evaluate the estimation methods by simulation studies. An application of the new distribution to a real data is provided for illustration purpose in which the proposed models represent the data better than the generalized standard arcsine distribution. We hope that the new model will be very useful in the fields of probability, statistics and other branches of applied science.

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Competing Interests

Author has declared that no competing interests exist.

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